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PRODUCTS OF REGULAR CARDINALS AND CARDINAL INVARIANTS OF PRODUCTS OF BOOLEAN ALGEBRAS

BY

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ABSTRACT

We answer some questions of Monk, and give some information on others concerning cardinal invariants of Boolean algebras under ultraproducts and products.

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†† §1–4 of this paper are essentially the letters which the author sent in December 1987 to Monk solving problems from his notes on cardinal invariants of B.A.; §8 and §9 were written for §4; the other sections, §§5, 6 and 7, were completed in March 1988. Concerning §§5–9, for further results see Abstracts of AMS and subsequent papers. §10 was written during the Arcata meeting, summer 1985, and §11 in January 1986, after questions of Todorcevic.

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The results in §§4–9 are substantially improved in [Sh 355], [Sh 371] and [Sh 400].

§1. On λ-c.c. in ultraproducts of Boolean algebras

We point out that

$(*)_{\lambda,\mu,\theta}$ *if* D is a filter on θ, for $i < \theta$, B_i is a λ-c.c. Boolean algebra, *then* $\prod_{i<\theta} B_i/D$ is a μ-c.c. Boolean algebra

is independent of ZFC,[†] and that λ^+-c.c. is not preserved by ultraproducts of countably many Boolean algebras.

Remember:

1.1. Definition. Let $\lambda \to [\mu]^n_{\kappa,\theta}$ *iff* for any $c : [\lambda]^n \to \kappa$ there is $A \in [\lambda]^\mu$ such that $|\text{Rang}(c \restriction [A]^n)| \leqq \theta$.

[†] Even fixing cardinal arithmetic.

Also $\lambda \to [\mu]^n_{\kappa,<\theta}$ is defined similarly.

We shall use the obvious monotonicity properties. By [Sh 288],†

1.2. Theorem. *If* $\lambda^{<\lambda} = \lambda = \mathrm{cf}\,\lambda < \mu$, μ *strongly Mahlo, then for some* λ^+*-c.c.* λ*-complete forcing notion* P *of power* μ,

$$\Vdash_P \text{“}2^\lambda = \mu \;\&\; \mu \to [\lambda^+]^2_{\sigma,2} \qquad \text{for } \sigma < \lambda\text{”}.$$

Now

1.2A. Claim. (1) *If* $\mu \to [\lambda]^2_{2^\theta,<\aleph_0}$, *then*:

$(*)_{\lambda,\mu,\theta}$ *if* D is a filter on θ, and B_i is a Boolean algebra satisfying the λ-c.c. for $i < \theta$, *then* $B = \Pi_{i<\theta} B_i/D$ satisfies the μ-c.c.

(2) We can replace 2^θ by $\mathrm{Min}\{|E| : E \subseteq D \text{ generates } D\}$.

Proof. Let, for $\alpha < \mu$, $a_\alpha = \langle a_i^\alpha : i < \theta\rangle/D \neq 0$, for $\alpha < \beta < \mu$, $B \vDash a_\alpha \cap a_\beta = 0$, and for $\alpha < \mu$, $B \vDash a_\alpha \neq 0$.

Let $c(\alpha, \beta) = \{i : a_i^\alpha \cap a_i^\beta = 0 \;\&\; a_i^\alpha \neq 0 \;\&\; a_i^\beta \neq 0\}$. So c is a coloring of μ, two place, $|\mathrm{Rang}\, c| \leqq 2^\theta$ (just the power of a set generating D is enough) and $c(\alpha, \beta) \in D$ for $\alpha < \beta < \mu$.

So on some $A \in [\mu]^\lambda$, $\mathrm{Rang}(c \restriction [A]^2)$ is finite; so the intersection $\bigcap \mathrm{Rang}(c \restriction [A]^2)$ is in D hence nonempty. So for some i $(\forall \alpha < \beta \text{ in } A)$ $[a_i^\alpha \cap a_i^\beta = 0 \;\&\; a_i^\alpha \neq 0 \;\&\; a_i^\beta \neq 0]$, so $\{a_i^\alpha : \alpha \in A\} \subseteq B_i$ shows B_i does not satisfy λ-c.c.; contradiction.

1.3. Conclusion. The question whether $(*)_{\lambda,\mu,\theta}$ holds does not depend on cardinal arithmetic alone.

Proof. Start e.g. with $V \vDash$ GCH. By 1.2, 1.2A we get one case: $(*)_{\lambda,\mu,\theta}$ holds. If we use P = adding μ Cohen subsets to λ we get $\neg(*)_{\lambda,\mu,\theta}$, but the same cardinal arithmetic.

1.3A. Claim. If $\mu \to [\lambda]^2_{\theta,<\kappa}$, μ regular for simplicity and for $i < \theta$, B_i is a Boolean algebra and B, the product of $\langle B_i : i < \theta\rangle$, does not satisfy the μ-c.c., *then* for some $a \subseteq \theta$, $|a| < \kappa$, the product of $\langle B_i : i \in a\rangle$ does not satisfy the λ-c.c.

Proof. Similar, so we leave it to the reader.

† For a weaker (but sufficient) result, see [Sh 276].

1.4. Theorem. *Suppose $\lambda > \aleph_1$ is regular. Then there are Boolean algebras B_n $(n < \omega)$ such that:*

(i) *B_n satisfies the λ-c.c.,*

(ii) *for any uniform ultrafilter D on ω (or filter containing the cobounded subsets) $\Pi_{n<\omega} B_n/D$ does not satisfy the λ-c.c. except possibly when*

(*) *λ is Mahlo and for every $\langle C_\mu : \mu < \lambda, \mu$ is inaccessible$\rangle$, C_μ a club of μ there is C a club of λ such that $\forall \alpha < \lambda\, \exists \mu (C \cap \alpha = C_\mu \cap \alpha)$.*

Remark. If λ is a successor or just not Mahlo then (*) fails trivially. Also if there are stationary $S_i \subseteq \lambda$ such that for any inaccessible $\lambda' < \lambda$ $(\exists i < \lambda')[S_i \cap \lambda'$ not stationary] then (*) fails (see [Sh 276], 3.9).

Proof. See [Sh 276] proof of 3.11, 3.3, §3 (which continues Todorcevic [T 2]). By the proof, for such λ, there is a symmetric function c from $[\lambda]^2$ to ω such that:

(A) *if* $n < \omega$, $i \leqq \zeta_i^1 < \zeta_i^2 < \cdots < \zeta_i^n$ for $i < \lambda$ and $m < \omega$, *then* for some $i < j$:

$$\zeta_i^n < \zeta_j^1 \quad \text{and} \quad \bigwedge_{l=1}^{n} \bigwedge_{k=1}^{n} c(\zeta_j^l, \zeta_i^k) \geqq m.$$

We define a Boolean algebra B_n: it is freely generated by $\{x_i^n : i < \lambda\}$ except:

$$B_n \vDash x_\alpha^n \cap x_\beta^n = 0 \qquad \text{when } \alpha < \beta \ \&\ c(\beta, \alpha) \leqq n.$$

Now $B_n \vDash \lambda$-c.c. by (A) (for each n) but $\Pi B_n/D \vDash \neg\lambda$-c.c. as

$$\langle \langle x_\alpha^n : n < \omega \rangle / D : \alpha < \lambda \rangle$$

exemplify this.

1.5. Conclusion. If $\lambda \geqq \aleph_1$, *then for some B_n $(n < \omega)$*

(i) $c(B_n) \leqq \lambda$,

(ii) $c(\Pi_{n<\omega} B_n/D) \geqq \lambda^+$ for every uniform ultrafilter D on ω.

1.6. Observation. If D is ultrafilter on I, $\lambda \to (\lambda_i)^2_{i\in I}$, $B_i \vDash \lambda_i$-c.c., then $\Pi B_i/D \vDash \lambda$-c.c.

§2. On length of Boolean algebras

2.1. Definition. For a Boolean algebra, B, let:

$$\text{Length}(B) = \sup\{|Y| : Y \subseteq B,\ Y \text{ is linearly ordered}\}.$$

We shall prove that the length of $\Pi_{i<\kappa} B_i$ cannot be computed from $\langle \mathrm{Length}(B_i) : i < \kappa \rangle$ alone.

2.2. LEMMA. (1) *Let* $T \subseteq {}^{\kappa \geqq}\lambda$ *be a tree (with* κ *levels)* [*i.e.* $\eta \in T \Rightarrow \{\eta \restriction \alpha : \alpha \leqq \lg(\eta)\} \subseteq T$] *and*

$$[\eta \in T \cap {}^{\alpha}\lambda, \alpha < \beta < \kappa \Rightarrow \exists^{>1} \nu \in T \cap {}^{\beta}\lambda (\nu \restriction \alpha = \eta)].$$

For each α *let* $T_\alpha = T \cap {}^{\alpha}\lambda$, $<_\alpha$*-lexicographic order on* T_α. *Let* B_α *be the interval Boolean algebra of* $(T_\alpha, <_\alpha)$. *Then*

(a) $\mathrm{Length}(B_\alpha) = |T_\alpha|$ *if* T_α *is infinite, and* $2^{|T_\alpha|} < \aleph_0$ *if* T_α *is finite*;

(b) $\mathrm{Length}(\Pi_{\alpha<\kappa} B_\alpha) \geqq |T_\kappa|$ ($\kappa \geqq \aleph_0$, *of course*).

(2) *Let* B'_α *be the interval Boolean algebra of the cardinal* $|T_\alpha|$. *Then*

(a) $\mathrm{Length}(B'_\alpha) = |T_\alpha|$ *if* T_α *is infinite, and* $2^{|T_\alpha|}$ $(< \aleph_0)$ *if* T_α *is finite*,

(b) $\mathrm{Length}(\Pi_{\alpha<\kappa} B'_\alpha) \leqq \mu \stackrel{\text{def}}{=} \Sigma_{\alpha<\kappa} |T_\alpha|^{\aleph_0} + 2^\kappa$ *when* κ *has uncountable cofinality*.

PROOF. (1)(a) Immediate.

(1)(b) W.l.o.g. $0_\alpha = \langle 0 : i < \alpha \rangle \in T_\alpha$, for $\eta \in {}^{\kappa}\lambda \cap T$ let $a_\eta = \langle [0_\alpha, \eta \restriction \alpha) : \alpha < \kappa \rangle \in \Pi_{\alpha<\kappa} B_\alpha$.

(2)(a) Immediate.

(2)(b) Let $\lambda \geqq \mu$, $\lambda = \lambda^{\aleph_0}$.

Let J be a linear order, $|J| > \lambda$ and suppose there are $a_t = \langle a_t^\alpha : \alpha < \kappa \rangle \in \Pi B'_i$ for $t \in J$ and $\langle a_t : t \in J \rangle$ a chain in $\Pi B'_i$. We shall get a contradiction thus finishing the proof.

Now for each α

(*) we can find $\langle A_n^\alpha : n < \omega \rangle$, $\langle m_n^\alpha : n < \omega \rangle$ and h_α^n such that:

$$B'_\alpha \setminus \{0\} = \bigcup_{n<\omega} A_n^\alpha,$$

$h_\alpha^n : A_n^\alpha \to {}^{m_n^\alpha}|T_\alpha|$ (sequence of length m_n^α of ordinals $< |T_\alpha|$) such that:

($\oplus$) if $c, d \in A_n^\alpha$, then the truth values of $c = d$, $c < d$ depend just on the equalities and inequalities between the ordinals in the sequences $h_n^\alpha(c)$, $h_n^\alpha(d)$.

As $\lambda \geqq 2^\kappa$, we know that w.l.o.g. for some $\langle n(\alpha) : \alpha < \kappa \rangle$, we have: $(\forall t \in J) a_t^\alpha \in A_{n(\alpha)}^\alpha$.

Now for every $A \subseteq \kappa$, $a_t \restriction A \stackrel{\text{def}}{=} \langle a_t^\alpha : \alpha \in A \rangle \in \Pi_{\alpha\in A} B'_\alpha$ is $\leqq$-increasing and $\{A \subseteq \kappa : |\{a_t \restriction A : t \in J\}| \leqq \lambda\}$ is an ideal of κ and, as $\lambda = \lambda^{\aleph_0}$, it is $\aleph_1$-complete.

So w.l.o.g. $n(\alpha)=n(*)$ (in the case $\{\alpha : n(\alpha)=n(*)\}$ is bounded in κ, we can redefine κ, and λ still satisfies the requirement).

Now we have that (w.l.o.g.):

(**) there is $n(*)<\omega$ such that for each $\alpha<\kappa$ we have: $\{a^\alpha_t : t\in J\}$ has ordertype a scattered set of rank $\leqq n(*)$ (the point is just that the $n(*)$ is fixed).

We get a contradiction by induction on $n(*)$ (simultaneously for all Boolean algebras and B_α, J, and a^α_t).

The case $n(*)=0$ is empty.

The case $n(*)>1$. There are convex[†] equivalence relations e_α on $\{a^\alpha_t : t\in J\}$ of order type $\leqq\lambda$ or $\leqq\lambda^*$ (= the inverse of λ) with each equivalence class scattered of rank $\leqq n(*)-1$.

Now e_α induces a convex equivalence relation e'_α on J, i.e. $t_1 e'_2 t_2$ iff $a^\alpha_{t_1} e_\alpha a^\alpha_{t_2}$; e'_α is convex as $[t_1\leqq t_2 \Rightarrow a^\alpha_{t_1}\leqq a^\alpha_{t_2}]$. Also $\bigcap_{\alpha<\kappa} e'_\alpha$ is a convex equivalence relation on J. Now each equivalence class I has power $\leqq\lambda$, otherwise we have $\langle a_t : t\in I\rangle$ and apply an induction hypothesis on $n(*)$. Now choose $J'\subseteq J$ which is a set of representatives for $\bigcap_{\alpha<\kappa} e'_\alpha$, i.e. such that for each $(\bigcap_\alpha e'_\alpha)$-equivalence class I we have $|J'\cap I|=1$. So necessarily $|J'|>\lambda$. Now we choose b^α_t for $\alpha<\kappa$, $t\in J'$ such that:

$$b^\alpha_t \in \{a^\alpha_s : s\in t/e'_\alpha\},$$

$$b^\alpha_{t_1}=b^\alpha_{t_2} \Leftrightarrow t_1 e'_\alpha t_2.$$

This is easy to do. Now apply our induction hypothesis to $(\langle b^\alpha_t : t\in J'\rangle : t\in J')$, $n(*)-1$.

Now we come to the main case.

The case $n(*)=1$. As we can replace $\Pi_{\alpha<\kappa} B_\alpha$ and $\langle a^\alpha_t : \alpha<\kappa, t\in J\rangle$ by $\Pi_{\alpha\in A} B_\alpha$ and $\langle a^t_\alpha : \alpha\in A, t\in J\rangle$ as long as $|\{a_t\restriction A : t\in J\}\|>\lambda$; and as we can replace the a^α_t's by $(1_{B_\alpha}-a^\alpha_t)$'s, w.l.o.g.

($\oplus$) $\langle a^\alpha_t : t\in J\rangle$ is well ordered of order type $\leqq |T_\alpha|=\lambda_\alpha$ (for each α).

So w.l.o.g. $J\subseteq \Pi_{\alpha<\kappa}\lambda_\alpha$ is ordered by $\eta\leqq\nu\Leftrightarrow \bigwedge_\alpha \eta(\alpha)\leqq\nu(\alpha)$. Let χ be regular large enough, $<^*_\chi$ a well order of $H(\chi)$ (the family of sets of hereditary power $<\chi$).

[†] An equivalence relation e on I is convex iff $\forall x\in I$ [x/e is a convex set].

Let $N_0 \prec (H(\chi),\in,<^*_\chi, J)$, $2^\kappa \subseteq N_0$, $[|N_0|]^\kappa \subseteq N_0$, $\| N_0 \| = 2^\kappa$. Let $N_0 \prec M \prec (H(\chi),\in,<^*_\chi, J)$, $\| M \| = \lambda$, $[|M|]^{\aleph_0} \subseteq M$, $\lambda + 1 \subseteq M$.

Let $\langle \langle \gamma^\delta_i : i < \mathrm{cf}\,\delta \rangle : \delta < \lambda \rangle$ be the $<^*_\chi$-first sequence such that $\langle \gamma^\delta_i : i < \mathrm{cf}\,\delta \rangle$ is increasing with limit δ. Choose $\eta \in J - |M|$ and define (for $N \prec M$, such that $\kappa + 1 \subseteq N$): $\rho_N(\eta) \in {}^\kappa |N|$, $[\rho_N(\eta)](\alpha) \stackrel{\mathrm{def}}{=} \mathrm{Min}\{\gamma \in N : \gamma \geqq \eta(\alpha)\}$. Note: if $\eta(\alpha) \notin N$ then $\mathrm{cf}[(\rho_N(\eta))(\alpha)]$ is a regular cardinal which belongs to N but is not included in it and is $> \kappa$. We choose by induction on n, $\zeta_n < \lambda$ as follows: letting $N_n = $ Skolem Hull$(N_0 \cup \{\zeta_0, \ldots, \zeta_{n-1}\})$, if

$$\{\mathrm{cf}[[\rho_{N_n}(\eta)](\alpha)] : [\rho_{N_n}(\eta)](\alpha) \notin N_n\}$$

is a singleton $\{\mu_n\}$ (or is empty and we let $\mu_n = \kappa$), we can choose $\zeta_n < \mu_n$ such that *if* $\alpha < \kappa$, $[\rho_{N_n}(\eta)](\alpha) \notin N_n$, *then*

$$\gamma^{\rho_{N_n}(\eta)(\alpha)}_{\zeta_n} > \eta(\alpha)$$

(see above on $\langle \langle \gamma^\delta_\zeta : \zeta > \delta \rangle \rangle$). First assume ζ_n is defined for each n.

So for every α, $\langle [\rho_{N_n}(\eta)](\alpha) : n \rangle$ decreases and stops only when $\rho_{N_n}(\eta)(\alpha) \in N_n$. So if we succeed in continuing a step, then $\bigwedge_\alpha \eta(\alpha) = \rho_{N_{k_\alpha}}(\eta)(\alpha) \in N_{k_\alpha}$ for some $k_\alpha < \omega$, so $\eta \subseteq$ the Skolem Hull of $N_0 \cup \{\zeta_n : n < \omega\}$. Of course, $\langle \zeta_n : n < \omega \rangle$ depends on η but there are $\leqq \lambda^{\aleph_0}$ such sequences, and

$$\| N_0 \cup \{\zeta_n : n < \omega\} \| \leqq 2^\kappa;$$

so for some $\eta \in J$, and n, $\zeta_0, \ldots, \zeta_{n-1}$ are defined but not ζ_n. So for this n $\{\mathrm{cf}[(\rho_{N_n}(\eta)](\alpha)) : \alpha < \kappa\}$ has more than one element, i.e. for some $\alpha_1, \alpha_2 < \kappa$:

$$\mu_1 \stackrel{\mathrm{def}}{=} \mathrm{cf}(\rho_{N_n}(\eta)(\alpha_1)) < \mu_2 \stackrel{\mathrm{def}}{=} \mathrm{cf}(\rho_{N_n}(\eta)(\alpha_2)).$$

Choose ζ^*, $\sup[N_n \cap \mu_1] < \zeta^* < \mu_1$, let

$$N^* = \text{Skolem Hull of } (N_n \cup \{\zeta^*\}).$$

So in N^*, there is ζ^* such that:

$$\sup[N_n \cap (\rho_{N_n}(\eta))(\alpha_1)] < \zeta^* < [\rho_{N_n}(\eta)](\alpha_1).$$

Now

(α) $\sup N^* \cap \mu_2 = \sup(N_n \cap \mu_2)$
(as $\mu_1, \mu_2 \in N_n$, $\mu_1 < \mu_2$ are regular).

(α)′ Similarly for

$$\sup[N^* \cap (\rho_{N_n}(\eta))(\alpha_2)] = \sup[N_n \cap (\rho_{N_n}(\eta))(\alpha_2)].$$

(β) $N_n \models (\forall(x)$ (if x is an ordinal $< [\rho_{N_n}(\eta)](\alpha_1)$ then there is $y \in J$, such that

$$x < y(\alpha_1) < [\rho_{N_n}(\eta)(\alpha_1)],$$

$$y(\alpha_2) < [\rho_{N_n}(\eta)(\alpha_2)).$$

Note: $[\rho_{N_n}(\eta)](\alpha_1)$, $[\rho_{N_n}(\eta)](\alpha_2)$ are in N_n, though not the function $\rho_{N_n}(\eta)$! Hence also N^* satisfies this formula; now apply it to $x = \gamma^{\delta}_{\zeta*}$ where $\delta = [\rho_{N_n}(\eta)](\alpha_1)$ to get $y = \nu$. So $\nu(\alpha_1) > \eta(\alpha_1)$ [choice of ζ^*], $\nu(\alpha_2) < \eta(\alpha_2)$ [as $\nu(\alpha_1) \in N^* \cap [\rho_{N_n}(\eta)](\alpha_2)$ and $(\alpha)'$]. This contradicts our assumption on J.

2.3. Conclusion. If e.g. $\lambda = \lambda^{\aleph_0}$, $\lambda^\kappa > \lambda + 2^\kappa$, *then* for some B_i, B'_i, $i < \kappa$,

$$\text{Length}(B'_i) = \lambda = \text{Length } B_i,$$

$$\text{Length}\left(\prod_{i<\kappa} B'_i\right) = \lambda < \lambda^\kappa = \text{Length}\left(\prod_{i<\kappa} B_i\right).$$

§3. On depth of Boolean algebras

3.1. Definition. The depth of a Boolean algebra is

$$\text{Dp}(B) \stackrel{\text{def}}{=} \sup\{|X| : X \text{ is well ordered}\}.$$

We shall show that, in some universes of set theory, $\text{Dp}(\Pi_{i<\kappa} B_i/D)$ is $</>$ $\Pi_{i<\kappa}(\text{Dp}(B_i))/D$ for some Boolean algebra B_i and ultrafilter D.

3.1A. Remark. $\text{Length}(\Pi_{i<\kappa} B_i/D) \geqq \Pi_{i<\kappa} \text{length}(B_i)/D$ for any ultrafilter D on κ, B_i Boolean algebras, by Łos theorem as observed by S. Koppleberg and the author independently.

Remark. Of course, for some regular ultrafilter D on λ, in ω^λ/D there is a decreasing sequence of length 2^λ (see e.g. [ShA 1, VI, NB]) so the problem is to find cases in which this fails; necessarily GCH cannot hold.

3.2. Theorem. *Suppose* CH, $\lambda > \aleph_1$, *P is the product of λ Sacks forcing with countable support*: $\Pi_{i<\lambda} Q_i$. *Then in* V^P:

(a) $2^{\aleph_0} = (\lambda^{\aleph_0})^V$,

(b) *for some ultrafilter D on ω (non-principal) in $(\omega, <)^\omega/D$ there is no increasing chain of length $\aleph_2$ (nor decreasing),*

(c) *if B_0 is atomless countable Boolean algebra, then in B_0^ω/D there is no increasing (nor decreasing) chain of length $\aleph_2$.*

Proof. By a theorem of Laver, there is an ultrafilter D on ω (non-principal) such that D is an ultrafilter also in V^P (more accurately — generates one). This is our D.

Let $p \in P$, $p \Vdash$ "$\langle \underset{\sim}{f_\alpha}/D : \alpha < \aleph_2 \rangle$ a counterexample". For each $\alpha < \aleph_2$, there is p_α, $p \leqq p_\alpha \in P$, such that above p_α, $\underset{\sim}{f_\alpha}$ $(\in {}^\omega\omega)$ (or $\in {}^\omega B_0$) is a name in $\Pi_{i \in I_\alpha} Q_i$, $I_\alpha \subseteq \lambda$, $|I_\alpha| = \aleph_0$, $p_\alpha \in \Pi_{i \in I_\alpha} Q_i$. As $\tilde{V} \vDash$ CH w.l.o.g. $\langle I_\alpha : \alpha < \omega_2 \rangle$ is a Δ-system with heart I, and $(I_\alpha, p_\alpha, \underset{\sim}{f_\alpha})$ for $\alpha < \aleph_2$ are pairwise isomorphic over I. For $\alpha < \beta$ we know $p \Vdash$ "$\underset{\sim}{f_\alpha}/D < \underset{\sim}{f_\beta}/D$" so there is $\underset{\sim}{A}_{\alpha,\beta} \in D$, w.l.o.g. from V such that

$$p \Vdash \text{“}\underset{\sim}{A}_{\alpha,\beta} \in D \wedge \bigwedge_{n \in \underset{\sim}{A}_{\alpha,\beta}} \underset{\sim}{f_\alpha}(n) < \underset{\sim}{f_\beta}(n)\text{”}.$$

We now know p_α, p_β are compatible (definition of P) so there is $p_{\alpha,\beta} \geqq p_\alpha, p_\beta$, $p_{\alpha,\beta} \in P$. W.l.o.g $p_{\alpha,\beta}$ force a value to $\underset{\sim}{A}_{\alpha,\beta}$, say $B_{\alpha,\beta}$. So $p_{\alpha,\beta} \Vdash$ "$\bigwedge_{n \in B_{\alpha,\beta}} \underset{\sim}{f_\alpha}(n) < \underset{\sim}{f_\beta}(n)$". Now every permutation of λ induces an automorphism of $P = \tilde{\Pi}_{i<\lambda} Q_i$; let h be such permutation mapping I_α onto I_β over I and interchanging $(p_\alpha, \underset{\sim}{f_\alpha})$ with $(p_\beta, \underset{\sim}{f_\beta})$. So $h(p_{\alpha,\beta}) \geqq h(p_\alpha), h(p_\beta)$ but $h(p_\alpha) = p_\beta$, $h(p_\beta) = p_\alpha$, etc., so

$$p \leqq h(p_{\alpha,\beta}) \Vdash \text{“} \bigwedge_{n \in B_{\alpha,\beta}} \underset{\sim}{f_\beta}(n) < \underset{\sim}{f_\alpha}(n)\text{”};$$

contradiction.

Remark. The argument is good for any antisymmetric relation.

3.3. **Theorem.** *Let $\lambda = \lambda^{<\lambda} < \mu = \mu^{<\mu}$ be such that $(\forall \kappa)[\kappa < \mu \Rightarrow \kappa^{<\lambda} < \mu]$, $\diamondsuit_{\{\delta<\mu^+ : \operatorname{cf}\delta = \mu\}}$, $\diamondsuit_\mu$. For a set I of ordinals we let*

$$Q_I = \{f : f \text{ a partial function from } I \text{ to } \lambda \text{ of power } < \lambda\},$$

order: inclusion.

(A) *In $V^{Q_{\mu^+}}$ there is a uniform regular ultrafilter D on λ such that:*

(a) *in $(\lambda, <)^\lambda/D$ there is no increasing chain of length μ^+,*

(b) *if $\mathfrak{B}$ is the Boolean algebra of finite cofinite subsets of λ* then *in $\mathfrak{B}^\lambda/D$ there is no increasing (or decreasing) sequence of length μ^+,*

(c) *in* (b) *we can let $\mathfrak{B}$ be any Boolean algebra $\mathfrak{B}$ (hence any partial order) of power λ.*

(B) *In $V^{Q_{\mu^+}}$, $\lambda = \lambda^{<\lambda}$, $2^\lambda = \mu^+$.*

Proof. Let

$\mathrm{ap}_I = \{\underset{\sim}{D} : \underset{\sim}{D}$ a Q_I-name of an ultrafilter (regular uniform) on λ s.t. for every α, $\underset{\sim}{D} \cap \mathscr{P}(\lambda)V^{Q_{I\cap\alpha}}$ is a $Q_{I\cap\alpha}$-name$\}$.

$\mathrm{AP} = \bigcup \{\mathrm{ap}_I : I \subseteq \mu^+$, and $|I| < \mu\}$, let $\alpha(\underset{\sim}{D})$ be the unique α such that $\underset{\sim}{D} \in \mathrm{ap}_\alpha$.

(1) Let $\alpha < \mu^+$. Let a type for $\underset{\sim}{D} \in \mathrm{ap}_\alpha$ be a pair (M, q) such that:

(i) M is a model in V, $|L(M)| + \| M \| \leqq \mu$;

(ii) q is a Q_α-name of a set of formulas (in say m-variables) over $M^\lambda/\underset{\sim}{D}$, finitely satisfiable in it (ultrapower in V^{Q_α}).

We may omit M.

(2) The type (M, q) is strongly omitted for $\underset{\sim}{D} \in \mathrm{ap}_\alpha$ *if* for $\gamma < \mu$, in $V^{Q_{\alpha+\gamma}}$, if we extended $\underset{\sim}{D}$ by $< \mu$ sets getting D' still for no $g \in V^{Q_{\alpha+\gamma}}$

$$\bigwedge_{\varphi \in q} [\{i < \lambda : M \models \varphi(g)(i)\} \in D']$$

[where all parameters of q are functions from λ to M, we compute their value at i].

3.3A. The Game Lemma. In the following game player I has a winning strategy:

it lasts μ^+ stages,

in stage α player I chooses $\underset{\sim}{D}_\alpha \in \mathrm{ap}_\alpha$ extending each $\underset{\sim}{D}_j$ $(j < i)$,

player II chooses a set Γ_α of types, each strongly omitted for $\underset{\sim}{D}_\alpha$.

In the end player I wins if, for $\underset{\sim}{D}_{\mu^+}$, each $(M, q) \in \Gamma_\alpha$ $(\alpha < \mu^+)$ is omitted.

Remark. We do not use $\diamondsuit_{\{\delta < \mu^+ : \mathrm{cf}\,\delta = \mu\}}$ for 3.3A.

Proof. By [Sh 162]. (For other applications and formulations see [Sh 107]; on a similar construction see [Sh 326], §3.)

3.3B. *The Game*$^+$. We can also demand on the $\underset{\sim}{D}_\alpha$ (from player I)

(*) if $I \subseteq \mu^+$, $\mathrm{cf}\,\alpha = \mu$, $|I| < \mu$, $\underset{\sim}{E}$ a Q_I-name of an ultrafilter, $\underset{\sim}{E} \restriction (I \cap \alpha) \subseteq \underset{\sim}{D}_\alpha$, *then* some order preserving $h : I \underset{\text{onto}}{\rightarrow} J \subseteq \alpha$ the h-image of $\underset{\sim}{E}$ is $\subseteq \underset{\sim}{D}_\alpha$, $h \restriction (I \cap \alpha) = \mathrm{id}_{I \cap \alpha}$.

[Hence $\underset{\sim}{D}_\alpha$ $(\mathrm{cf}\,\alpha = \mu)$ is a good ultrafilter.]

Proof of the Theorem 3.3. Let $\mathfrak{B}$ be a fixed order of power λ of order type $\zeta + 1$ or $(\zeta + 1)^*$. Build $\underset{\sim}{D}_\alpha \in \mathrm{ap}_\alpha$ increasing with α, by induction on α according to the winning strategy of the game of 3.3A.

In stage δ, $\mathrm{cf}\,\delta = \mu$, $\diamondsuit_{\{\delta < \mu^+ : \mathrm{cf}\,\delta = \mu\}}$ gives us the guess $\langle \underset{\sim}{f}^\delta_\alpha / \underset{\sim}{D}_\delta : \alpha < \delta \rangle$ which is (forced to be) $<_{D_\delta}$-increasing, $\underset{\sim}{f}^\delta_\alpha$ a $\underset{\sim}{Q}_\alpha$-name of a function from λ to $\mathfrak{B}$. Now we define (M, q):

$$M = \mathfrak{B},$$

$$\underset{\sim}{q} = \{\underset{\sim}{f}^\delta_\alpha / \underset{\sim}{D}_\delta \leqq x/D_\delta : \alpha < \delta\}$$
$$\cup \left\{ x/D_\delta \leqq h/\underset{\sim}{D}_\delta : h \in ({}^\lambda\mathfrak{B})^{V^{Q_\alpha}} \text{ and } \bigwedge_{\alpha<\delta} f^\delta_\alpha/\underset{\sim}{D}_\delta < h/\underset{\sim}{D}_\delta \right\}$$

(remember that $\underset{\sim}{q}$ is a Q_δ-name [and if $\langle f^\delta_\alpha/\underset{\sim}{D}_\delta : \alpha < \delta\rangle$ is $<_{D_\delta}$-decreasing, we invert the order of $\mathfrak{B}$ and continue similarly; so we ignore this].

We should prove that it is strongly omitted; so we let $G \subseteq Q_\delta$ be generic over V and work in $V[G]$. Let $\gamma < \mu$ and $\underset{\sim}{D}'$ be generated by $D_\delta \cup \{\underset{\sim}{A}_i : i < i(*) < \mu\}$ where $\underset{\sim}{A}_i$ is a $Q_{\delta+\gamma}/G$-name.

So assume $\underset{\sim}{g}$ is a $Q_{\delta+\lambda}/G$-name, $p \in Q_{\delta+\lambda}/G$, $p \Vdash$ "$\underset{\sim}{g}, \{\underset{\sim}{A}_i : i < i(*) < \mu\}$ is a counterexample and w.l.o.g. $\{\underset{\sim}{A}_i : i < i(*) < \mu\}$ is closed under finite intersection". So for each $\alpha < \delta$ there are $p_\alpha, j(\alpha)$ such that:

(a) $p \leqq p_\alpha \in Q_{\alpha+\lambda}/G$,

(b) $p_\alpha \Vdash$ "$[\{i : f^\delta_\alpha(i) < \underset{\sim}{g}(i)\} \supseteq \underset{\sim}{A}_j(\alpha) \cap B_\alpha, B_\alpha \in D, j(\alpha) < i(*)$".

So for some unbounded $Z \subseteq \delta$, for $\alpha \in Z$, $p_\alpha = p^*$, $j(\alpha) = j(*)$ (or really $p_\alpha \restriction [\alpha, \alpha+\gamma] = p^*$).

Now for each $i < \lambda$ let $T_i \subseteq \mathfrak{B}$ be the set of $b \in \mathfrak{B}$ such that:

$$p^* \nVdash \neg\, [\underset{\sim}{g}(i) = b \wedge i \in \underset{\sim}{A}_{j(*)}].$$

Clearly $A^* = \{i : T_i \neq \emptyset\} \in D$. And by (b) above

$$\alpha \in Z \;\&\; i \in A^* \cap B_\alpha \;\&\; b \in T_\alpha \Rightarrow \mathfrak{B} \vDash f^\delta_\alpha(i) \leqq b.$$

So for $\alpha \in Z$:

$$(*) \qquad \langle b_i : i < \lambda\rangle \in (\Pi T_i)^{V^{Q_\delta}} \Rightarrow f^\delta_\alpha/D \leqq \langle b_i : i < \lambda\rangle/D.$$

Remember $\mathfrak{B}$ is $\zeta + 1$ or $(\zeta + 1)^*$, $\zeta < \lambda^+$. So $\mathfrak{B}$ is a well ordering (linear) or inverse well ordering with minimal element. Let $b_i = \inf T_i$, then

$$\bigwedge_\alpha f^\delta_\alpha/D \leqq \bar{b}/D \qquad \text{where } \bar{b} = \langle b_i : i < \lambda\rangle \in ({}^\lambda\lambda)^{V^{Q_\alpha}}$$

and so $x/D < \bar{b}/D \in q$, but this is impossible. This proves 3.3(A)(a).

End of the Proof of 3.3(A)(b). Let $\mathfrak{B}$ be the finite cofinite subsets of λ; if in $\mathfrak{B}^\lambda/D$ there is a monotonic sequence $\langle f_i/D : \alpha < \mu^+\rangle$ then w.l.o.g. it is increasing (otherwise use $1_{\mathfrak{B}} - f_\alpha/D$) and w.l.o.g. $\{i : f_\alpha(i) \text{ is finite}\} \in D$ for each

$\alpha < \lambda$ (if it fails for α_0 use $\langle f_{\alpha_0+\alpha}/D - f_\alpha/D : \alpha < \lambda \rangle$), hence w.l.o.g. $f_\alpha(i)$ is finite for $\alpha < \mu^+$, $i < \lambda$; let $f^*_\alpha(i) = |f_\alpha(i)|$, hence $\langle f^*_\alpha/D : \alpha < \lambda \rangle$ is strictly monotonic and we get a contradiction.

Proof of 3.3(A)(c). Use the $<$-system density for μ^+ which we are allowed to use (see [Sh 162]) and the symmetry in the forming.

3.4. Conclusion. For the forcing notion from 3.3: in $V^{Q_{\mu^+}}$, D is a regular ultrafilter on λ (even good) and $\mathfrak{B}$ the Boolean algebra we have

$$\lambda = \text{Depth } B \text{ (obtained)}; \qquad \Pi(\text{Depth } B)/D = \mu^+, \quad \text{Depth}(\Pi B/D) \leqq \mu.$$

3.5. Remark. (1) The property of the order $\mathfrak{B}$ we really use in the proof of 3.3(A)(a) is that it is complete not only in V but even in $V^{Q_{\mu^+}}$.

(2) Instead of μ^+ we can get an inaccessible 2^λ. E.g. if μ is strongly inaccessible Mahlo, $\lambda = \lambda^{<\lambda} < \mu$; force with

$R = \{\underset{\sim}{D}$: for some $I \subseteq \mu$, $(\forall \kappa)[\lambda < \kappa < \mu$ & κ strongly inaccessible $\Rightarrow |I \cap \kappa| < \kappa]$ and $\underset{\sim}{D}$ is a Q_I-name of regular uniform ultrafilter on λ such that for every α, $\underset{\sim}{D} \cap \mathscr{P}(\lambda)^{V^{Q_I \cap \alpha}}$ is a $Q_I \cap \alpha$-name$\}$.

§4. Spread and entangled orders

4.1. Definition. For a Boolean algebra B let $s(B)$, the spread of B, be

(*) $s(B) \overset{\text{def}}{=} \sup\{|Y| : Y \subseteq B - \{0\}$ and no $y \in Y$ belongs to the ideal generated by $Y - \{y\}\}$

or equivalently

(*)′ $s(B) = \sup\{c(B') : B'$ is a homomorphic image of $B\}$ [where $c(B')$ is the cellularity number of B'].

4.2. Problem. So we have, for $\lambda = s(B)$ a limit cardinal, two attainment problems:

A. *Obtainment*. If $s(B) = \lambda$, is there $Y \subseteq B - \{0\}$, $|Y| = \lambda$ as in (*)?

B. *Weak Obtainment*. If $s(B) = \lambda$, is there a homomorphic image B' of B such that $c(B') = \lambda$?

Note that by [Sh 233]:

4.2C. THEOREM. *If $s(B)$ is singular and not obtained, then* $2^{\mathrm{cf}[s(B)]} > s(B)$.

So the obtainment problem for singular $\mu = s(B)$ is only for the case $2^{\mathrm{cf}\,\mu} > \mu$.

Todorcevic (see Monk [M]) proves that for $\lambda = 2^{\aleph_0}$, we can construct a Boolean algebra B with non-weak obtainment for $s(B) = \lambda$ (if $2^{\aleph_0}$ is a limit cardinal).

The problem of getting examples for non-obtainment is closely tied in with entangled linear orders and related properties (on these see Todorcevic [T 1]) which has a long historical discussion; see Abraham–Rubin–Shelah [ARS 153] and Bonnet–Shelah [BoSh 210].

Our main conclusion is 4.15.

4.3. OBSERVATION. If $s(A)$ (the spread) is singular and not obtained, *then* A has no homomorphic image B such that $c(B) = s(A)$, i.e. $s(A)$ is not weakly obtained.

PROOF. If for some homomorphic image B of A, $c(B) = s(A)$, *then* B has an antichain of power $s(A)$ (by the Erdős–Tarski theorem) hence $s(A)$ is obtained.

4.4. OBSERVATION. (1) If $s(A)$ (the spread) is not obtained and is strongly inaccessible, *then* for some homomorphic image B of A, $c(B) = s(A)$; in fact we have $B = A$.

(2) If λ is inaccessible, then there is a Boolean algebra B such that $c(B) = \lambda$ is not obtained.

PROOF. (1) If $c(A) = s(A)$, we finish. If not, $c(A) < s(A)$ hence $(\forall \mu < s(A))\mu^{c(A)} < s(A)$ so (as necessarily $|A| \geqq s(A)$; see [Sh 92]) A has an independent subset of cardinality $s(A)$ so $s(A)$ is *obtained*; contradiction.

(2) Well known.

4.5. REMARK. We can conclude that the double problem of being obtained is really double only for weakly inaccessibles.

4.6. DEFINITION. (1) $\mathrm{Ens}(\lambda, \mu, \kappa)$ means: there are linear orderings $\langle I_\alpha : \alpha < \kappa \rangle$ such that:

(a) I_α is a linear order of power λ,

(b) *if* $n < \omega$, $\alpha_1 < \cdots < \alpha_n < \kappa$, $w \subseteq \{1, \ldots, n\}$, $t^l_\zeta \in I_{\alpha_l}$ for $\zeta < \mu$, $l = 1, \ldots, n$ and $[\zeta_1 \neq \zeta_2 \Rightarrow t^l_{\zeta_1} \neq t^l_{\zeta_2}]$, *then* for some $\zeta < \xi < \mu$,

$$[l \in w \Rightarrow t^l_\zeta < t^l_\xi],$$

$$[1 \leqq l \leqq n \wedge l \notin w \Rightarrow t^l_\zeta > t^l_\xi].$$

(2) $\mathrm{Ens}_k(\lambda, \mu, \kappa)$ is defined similarly but $n \leqq k$.

(3) If we omit μ, this means $\lambda = \mu$.

(4) A linear order I is (μ, n)-entangled *if*: for every pairwise distinct $t^l_\zeta \in I$ $(1 \leqq l \leqq n, \zeta < \mu)$ such that $t^1_\zeta < t^2_\zeta < \cdots < t^n_\zeta$ and $w \subseteq \{1, \ldots, n\}$, *there are* $\zeta < \xi < \mu$ such that:

(*) $1 \leqq l \leqq n \Rightarrow [l \in w \Leftrightarrow t^l_\zeta < t^l_\xi]$.

(5) We omit μ if $|I| = \mu$; we omit n if it holds for all $n < \omega$.

4.7. FACT. (1) $\langle I \rangle$ witnesses $\mathrm{Ens}(\lambda, \mu, 1)$ *iff* I is a linear order of power λ, with no monotonic sequence of length μ.

(2) $\langle I, J \rangle$ witnesses $\mathrm{Ens}(\lambda, \mu, 2)$ *iff* I, J are linear orders of power λ, with no monotonic sequence of length μ, and I, J are μ-far (i.e. have no isomorphic subsets of power μ) and I, J^* are μ-far where J^* is the reverse order on J.

(3) If I has density $< \mu$, $\mu = \mathrm{cf}\,\mu$, *then* in the definition (4.6(4),(5)) of "I is μ-entangled" we can add:

(*)′ $t^l_\zeta < t^{l+1}_\xi$, $t^l_\xi < t^{l+1}_\zeta$ for $l = 1, \ldots, n-1$.

(4) If $n \geqq 2$, I is (μ, n)-entangled, *then* I has density $< \mu$.

(5) If I is μ-entangled, I has κ-pairwise disjoint intervals each of power λ, then $\mathrm{Ens}(\lambda, \mu, \kappa)$.

PROOF. (3) Let $J \in [I]^{<\mu}$ be dense in I. Suppose that $\langle \langle t^l_\zeta : l = 1, \ldots, n \rangle : \zeta < \mu \rangle$ is as in 4.6(4), (5). For each $l \in \{1, \ldots, n\}$, $t^l_\zeta < t^{l+1}_\zeta$, and so there exists $s^l_\zeta \in J$ such that $t^l_\zeta \leqq s^l_\zeta \leqq t^{l+1}_\zeta$ (and at least one inequality is strict). Define functions h_0, h_1 on μ by:

$$h_0(\zeta) := \langle s^1_\zeta, \ldots, s^{n-1}_\zeta \rangle,$$

$$h_1(\zeta) := \langle \langle \mathrm{TV}(t^l_\zeta = s^l_\zeta), \mathrm{TV}(t^{l+1}_\zeta = s^l_\zeta) \rangle : l = 1, \ldots, n \rangle$$

(where TV(–) is the truth value of –). $\mathrm{dom}(h_0) = \mu$ and $|\mathrm{Rang}(h_0)| \leqq |J|^{n-1} < \mu$. Similarly for h_1. Since $\mathrm{cf}(\mu) = \mu$, there exists $A \in [\mu]^\mu$ such that $h_0 \restriction A$ and $h_1 \restriction A$ are constant. That's to say, for some $s^1, \ldots, s^{n-1}$ in J, $\forall l \in \{1, \ldots, n-1\}$, $\forall \zeta \in A$,

$$t^l_\zeta \leqq s^l = s^l_\zeta \leqq t^{l+1}_\zeta.$$

Since the t_ζ^l are given as pairwise distinct, using $h_1 \restriction A$, one finds that

$$t_\zeta^l < s^l < t_\zeta^{l+1}.$$

W.l.o.g. $A = \mu$ (relabelling); now applying 4.6(4), there exists $\zeta < \xi < \mu$ such that $1 \leqq l \leqq n \Rightarrow [l \in w \Leftrightarrow t_\zeta^l < t_\xi^l]$, and in addition, for $l = 1, \ldots, n-1$,

$$t_\zeta^l < s_\zeta^l = s^l = s_\xi^l < t_\xi^{l+1} \quad \text{and} \quad t_\xi^l < s_\xi^l = s^l = s_\zeta^l < t_\zeta^{l+1}$$

so that $(*)'$ holds.

(4) E.g. $n = 2$.

Suppose that I has density at least μ. By induction on $\zeta < \mu$, choose t_ζ^1, t_ζ^2 such that:

(i) $t_\zeta^1 < t_\zeta^2$,

(ii) $t_\zeta^1, t_\zeta^2 \notin \{t_\xi^1, t_\xi^2 : \xi < \zeta\}$,

(iii) $(\forall \xi < \zeta)(\forall l \in \{1, 2\})(t_\zeta^1 < t_\xi^l \Leftrightarrow t_\zeta^2 < t_\xi^l)$.

Continue to define for as long as possible.

There are two possible outcomes.

Outcome (a): one gets stuck at some $\zeta < \mu$. Define $J := \{t_\xi^1, t_\xi^2 : \xi < \zeta\}$. So $(\forall t^1 < t^2 \in I - J)(\exists s \in J)(t^1 < s \not\Leftrightarrow t^2 < s)$. Since $t^1, t^2 \notin J$, it follows that $t^1 < s \wedge t^2 > s$ or $t^1 > s \wedge t^2 < s$. So J is dense in I and is of power $2|\zeta| < \mu$ — a contradiction.

Outcome (b): one can define t_ζ^1, t_ζ^2 for every $\zeta < \mu$. Then $\langle t_\zeta^1, t_\zeta^2 : \zeta < \mu \rangle$, $w = \{1, 2\}$ constitute an easy counterexample to the $(\mu, 2)$-entangledness of I.

4.8. Fact. For a linear order I and regular uncountable cardinal μ, the following are equivalent:

(a) I is μ-entangled.

(b) $B = \mathrm{BA}_{\mathrm{inter}}(I)$ (the interval Boolean algebra) is μ-narrow, i.e. with no μ pairwise incomparable elements.

Proof. (a)$\Rightarrow$(b). By 4.7(4) I has density $< \mu$.

Let $\langle \tau_\zeta : \zeta < \mu \rangle$ be distinct elements of B. We know that for each ζ there are an even $n(\zeta) < \omega$ and $t_\zeta^1 < \cdots < t_\zeta^{n(\zeta)}$ in I such that $\tau_\zeta = \bigcup_{l=1}^{n(\zeta)/2} [t_\zeta^{2l-1}, t_\zeta^{2l})$. As $\mathrm{cf}\,\mu > \aleph_0$, w.l.o.g. $n(\zeta) = n(*)$; now by 4.6(4) and 4.7(3) for some $\zeta < \xi$, for $l = 1, \ldots, n(*)/2$, $t_\zeta^{2l-1} \leqq t_\xi^{2l-1} < t_\xi^{2l} \leqq t_\zeta^{2l}$, hence $B \vDash \tau_\xi \subseteq \tau_\zeta$ as required.

(b)$\Rightarrow$(a). Note that I has density $<\mu$.[†]

So let $I_0 \subseteq I$ be a dense subset of I of cardinality $<\mu$. Let for $J \subseteq I$, $s < t$, from J, $(s,t)_J = \{r \in J : s < r < t\}$. Let

$$J = \{t \in I : \text{if } I \vDash s < t \text{ then } |(s,t)_I| = \mu \text{ and if } I \vDash t < s \text{ then } |(t,s)_I| = \mu\}.$$

Clearly

$(*)_1$ $|I - J| < \mu$ and if $s < t$ are in J then $|\{r \in J : s < r < t\}| = \mu$.

[why?

(a) If $|I - J| = \mu$, let $t_\zeta \in I - J$ be distinct for $\zeta < \mu$, so for each ζ there is $s_\zeta \in I$ such that $s_\zeta < t_\zeta$ & $|(s_\zeta, t_\zeta)_I| < \mu$ or $t_\zeta < s_\zeta$ & $|(t_\zeta, s_\zeta)_I| < \mu$. We can replace $\{t_\zeta : \zeta < \mu\}$ by any subset of the same cardinality so w.l.o.g. $s_\zeta < t_\zeta \Leftrightarrow s_0 < t_0$. By symmetry assume $s_0 < t_0$, otherwise look at I^*. For each ζ, as I_0 is a dense subset of I there is $r_\zeta \in I_0$ such that $s_\zeta \leqq r_\zeta \leqq t_\zeta$. As $|I_0| < \mu = \mathrm{cf}(\mu)$ w.l.o.g. $r_\zeta = r$ for every ζ. As $|[r_\zeta, t_\zeta]_I| \leqq |(s_\zeta, t_\zeta)_I| + 2 < \mu$ for each ζ, $|\{\xi < \mu ; t_\xi \leqq t_\zeta\}| \leqq |[r_\zeta, t_\zeta]_I| < \mu$. Clearly there is $h(\zeta) < \mu$ such that $[\xi < \mu$ & $\xi \geqq h(\zeta) \Rightarrow t_\zeta < t_\xi]$ and $C = \{\xi < \mu : (\forall \zeta < \xi) h(\zeta) < \xi\}$ is a club of μ, so $\langle t_\zeta : \zeta \in C \rangle$ is strictly increasing, contradicting "I has density $<\mu$".

(b) $s < t$ are in $J \Rightarrow |(s,t)_J| = \mu$ because $t \in J$ implies $\mu \leqq |(s,t)_I| \leqq |(s,t)_J| + |I \setminus J|$, but $|I \setminus J| < \mu$ so $\mu = |(s,t)_J|$.]

$(*)_2$ There is a dense subset J_0 of J of cardinality $<\mu$ [even easier].

Now let $t^l_\zeta \in I$ be distinct for $\zeta < \mu$, $l = 1, \ldots, n$ and $w \subseteq \{1, \ldots, n\}$ and we should find $\zeta < \xi$ such that:

$$[l \in w \Rightarrow t^l_\zeta < t^l_\xi], \quad [l \in \{1, \ldots, n\} \setminus w \Rightarrow t^l_\zeta > t^l_\xi].$$

We, of course, can replace $\{(t^1_\zeta, \ldots, t^n_\zeta) : \zeta < \mu\}$ by any subset of cardinality μ. So w.l.o.g.

$(*)_3$ no t^l_ζ is first or last, and every t^l_ζ is in J (as $|I - J| < \mu$).

So for each ζ we can find $r^1_\zeta, \ldots, r^{n+1}_\zeta \in J_0$ such that

[†] [I has no well-ordered subset of power μ nor an inverse well-ordered subset of power μ. So if I has density $\geqq \mu$, then there are disjoint closed–open intervals I_0, I_1 with density $\geqq \mu$. Now for each I_m we choose by induction on $\zeta <$ density(I_m) $a^m_\zeta < b^m_\zeta$ from I_m such that $[a^m_\zeta, b^m_\zeta]$ is disjoint from $\{a^m_\xi, b^m_\xi : \xi < \zeta\}$. So $\xi < \zeta \Rightarrow [a^m_\xi, b^m_\xi] \not\subseteq [a^m_\zeta, b^m_\zeta]$. Now $\langle [a^0_\zeta, b^0_\zeta) \cup (I_1 - [a^1_\zeta, b^1_\zeta)) : \zeta < \mu \rangle$ shows B is not μ-narrow.]

$$r^1_\zeta < t^1_\zeta < r^2_\zeta < t^2_\zeta < \cdots < t^n_\zeta < r^{n+1}_\zeta.$$

As $|I_0| < \mu = \mathrm{cf}(\mu)$ w.l.o.g. $r^l_\zeta = r_l$ for every l.

Let for each $\zeta < \mu$,

$$u_\zeta \stackrel{\text{def}}{=} \{l : l \in \{1, \ldots, n\} \text{ and } t^l_{2\zeta} < t^l_{2\zeta+1}\}.$$

u_ζ has $\leqq 2^n$ possible values. W.l.o.g. $u_\zeta = u^*$ for every $\zeta < \mu$.

Note $[l \notin u_\zeta \;\&\; l \in \{1, \ldots, n\} \Rightarrow t^l_{2\zeta} > t^l_{2\zeta+1}]$ (as $t^l_{2\zeta} \neq t^l_{2\zeta+1}$). For each $\zeta < \mu$, $l \in \{1, \ldots, n\}$ there is $p^l_\zeta \in J_0$ such that $t^l_{2\zeta} < p^l_\zeta < t^l_{2\zeta+1}$ or $t^l_{2\zeta+1} < p^l_\zeta < t^l_{2\zeta}$. W.l.o.g. $p^l_\zeta = p_l$.

Now we define by induction on $\zeta < \mu$, for every $l = \{1, \ldots, n\}$, members $q^{l,1}_\zeta, q^{l,2}_\zeta, q^{l,3}_\zeta, q^{l,4}_\zeta$ of J such that:

(i) if $l \in u_\zeta$ (i.e. $t^l_{2\zeta} < t^l_{2\zeta+1}$) then

$$r_l < q^{l,1}_\zeta < t^l_{2\zeta} < q^{l,2}_\zeta < p_l < q^{l,3}_\zeta < t^l_{2\zeta+1} < q^{l,4}_\zeta < r_{l+1};$$

(ii) if $l \notin u_\zeta$ (but $l \in \{1, \ldots, n\}$, i.e. $t_{2\zeta} > t^l_{2\zeta+1}$) then

$$r_l < q^{l,1}_\zeta < t^l_{2\zeta+1} < q^{l,2}_\zeta < p_l < q^{l,3}_\zeta < t^l_{2\zeta} < q^{l,4}_\zeta < r_{l+1};$$

(iii) $q^{l,m}_\zeta$ ($m \in \{1, 2, 3, 4\}$) does not belong to

$$\{q^{k,i}_\xi : \xi < \zeta, k \in \{1, \ldots, n\}, i \in \{1, \ldots, 4\}\} \cup \{t^l_\xi : \xi < \zeta, l \in \{1, \ldots, n\}\}.$$

There are no problems by $(*)_1$. It is still possible that for some $\zeta < \xi$,

$$\emptyset \neq \{q^{l,m}_\zeta : l = 1, \ldots, n \text{ and } m = 1, 2, 3, 4\} \cap \{t^l_\xi : l = 1, \ldots, n\}$$

for each ζ, there are at most $4n$ such ξ's, so there is $h_1(\zeta) < \mu$ such that $h_1(\zeta) \leqq \xi < \mu \Rightarrow \bigwedge_{l,m} \bigwedge_k q^{l,m}_\zeta \neq t^k_\xi$. So w.l.o.g.

$(*)_4$ the sets $\{q^{l,m}_\zeta, t^l_\zeta : l = 1, \ldots, n \text{ and } m = 1, 2, 3, 4\}$ are pairwise disjoint.

Now we define for every $\zeta < \mu$ a sequence $\langle s^l_\zeta : l = 1, \ldots, 4n \rangle$ by defining $s^{4l-1}_\zeta, s^{4l-2}_\zeta, s^{4l-3}_\zeta, s^{4l}_\zeta$ for each $l \in \{1, \ldots, n\}$ as follows:

Case 1. $l \in w, l \in u^*$,

$$\langle s^{4l-3}_\zeta, s^{4l-2}_\zeta, s^{4l-1}_\zeta, s^{4l}_\zeta \rangle = \langle t^l_{2\zeta}, q^{l,2}_\zeta, q^{l,3}_\zeta, t^l_{2\zeta+1} \rangle.$$

Case 2. $l \notin w, l \in u^*$,

$$\langle s^{4l-3}_\zeta, s^{4l-2}_\zeta, s^{4l-1}_\zeta, s^{4l}_\zeta \rangle = \langle q^{l,1}_\zeta, t^l_{2\zeta}, t^l_{2\zeta+1}, q^{l,4}_\zeta \rangle.$$

Case 3. $l \in w, l \notin u^*$,

$$\langle s_\zeta^{4l-3}, s_\zeta^{4l-2}, s_\zeta^{4l-1}, s_\zeta^{4l} \rangle = \langle q_\zeta^{l,1}, t_{2\zeta+1}^l, t_{2\zeta}^l, q_\zeta^{l,4} \rangle.$$

Case 4. $l \notin w, l \notin u^*$,

$$\langle s_\zeta^{4l-3}, s_\zeta^{4l-2}, s_\zeta^{4l-1}, s_\zeta^{4l} \rangle = \langle t_{2\zeta+1}^l, q_\zeta^{l,2}, q_\zeta^{l,3}, t_{2\zeta}^l \rangle.$$

Clearly for $\zeta < \mu$, $s_\zeta^1 < \cdots < s_\zeta^n$ and the s_ζ^l are pairwise distinct (by $(*)_4$) and

$$r_1 < s_\zeta^1 < s_\zeta^2 < p_1 < s_\zeta^3 < s_\zeta^4 < r_2 < s_\zeta^5 < s_\zeta^6 < p_2 < s_\zeta^7 < s_\zeta^8 < r_3 < \cdots.$$

Now for each ζ we define an element x_ζ of the Boolean algebra BA(I):

$$x_\zeta = \bigcup_{l=1}^{2n} [s_\zeta^{2l-1}, s_\zeta^{2l}).$$

Note

$(*)_5$ for $l = 1, \ldots, n$:
(a) $x_\zeta \cap [r_l, p_l) = [s_\zeta^{4l-3}, s_\zeta^{4l-2})$,
(b) $x_\zeta \cap [p_l, r_{l+1}) = [s_\zeta^{4l-1}, s_\zeta^{4l})$.

So $\langle x_\zeta : \zeta < \mu \rangle$ is a sequence of μ members of the Boolean algebra BA(I). By the assumption (we prove (b)$\Rightarrow$(a) in Fact 4.9) for some $\zeta < \xi < \mu$, x_ζ, x_ξ are comparable members of BA(I); i.e. $x_\zeta \subseteq x_\xi$ or $x_\xi \subseteq x_\zeta$. We derive our desired conclusion ($\otimes$) according to the case.

Case A. $x_\zeta \subseteq x_\xi$.

In this case we shall prove that $2\zeta + 1, 2\xi + 1$ are the ordinals we are looking for; i.e. conditions (α), (β), (γ) below hold, and we shall check those, thus finishing this case.

Condition α. $2\zeta + 1 < 2\xi + 1$.
[Trivial by $\zeta < \xi$.]

Condition β. If $l \in w$ then $t_{2\zeta+1}^l < t_{2\xi+1}^l$.

Possibility β1. $l \in u^*$.
Then $t_{2\zeta+1}^l = s_\zeta^{4l}$, $t_{2\xi+1}^l = s_\xi^{4l}$ (check the definition of the s's); now by $(*)_5$(b):

$$x_\zeta \cap [p_l, r_{l+1}) = [s_\zeta^{4l-1}, s_\zeta^{4l}),$$

hence (case 1 above)

$$x_\zeta \cap [p_l, r_{l+1}) = [q_\zeta^{l,3}, t_{2\zeta+1}^l);$$

and

$$x_\xi \cap [p_l, r_{l+1}) = [s_\xi^{4l-1}, s_\xi^{4l}),$$

hence (case 1 above)

$$x_\xi \cap [p_l, r_{l+1}) = [q_\xi^{l,3}, t^l_{2\xi+1}).$$

But as we are in Case A, $x_\zeta \subseteq x_\xi$ hence $x_\zeta \cap [p_l, r_{l+1}) \subseteq x_\xi \cap [p_l, r_{l+1})$, which means by the previous sentence $[q_\zeta^{l,3}, t^l_{2\zeta+1}) \subseteq [q_\xi^{l,3}, t^l_{2\xi+1})$, which implies $q_\xi^{l,3} \leqq q_\zeta^{l,3}$ and $t^l_{2\zeta+1} \leqq t^l_{2\xi+1}$. But $t^l_{2\zeta+1} \neq t^l_{2\xi+1}$ (as $\zeta \neq \xi$) so $t^l_{2\zeta+1} < t^l_{2\xi+1}$ as required.

Possibility β2. $l \notin u^*$.

Then $t^l_{2\zeta+1} = s_\zeta^{4l-2}$, $t^l_{2\xi+1} = s_\xi^{4l-2}$ (check the definition of the s's); now by $(*)_5$(a):

$$x_\zeta \cap [r_l, p_l) = [s_\zeta^{4l-3}, s_\zeta^{4l-2}),$$

hence (by case 3 above)

$$x_\zeta \cap [r_l, p_l) = [q_\zeta^{l,1}, t^l_{2\zeta+1});$$

and

$$x_\xi \cap [r_l, p_l) = [s_\xi^{4l-3}, s_\xi^{4l-2}),$$

hence (by case 3 above)

$$x_\xi \cap [r_l, p_l) = [q_\xi^{l,1}, t^l_{2\xi+1}).$$

But as we are in Case A, $x_\zeta \subseteq x_\xi$ hence $x_\zeta \cap [r_l, p_l) \subseteq x_\xi \cap [r_l, p_l)$, which means by the previous sentence $[q_\zeta^{l,1}, t^l_{2\zeta+1}) \subseteq [q_\xi^{l,1}, t^l_{2\xi+1})$, which implies $q_\zeta^{l,1} \geqq q_\xi^{l,1}$ and $t^l_{2\zeta+1} \leqq t^l_{2\xi+1}$. But $t^l_{2\zeta+1} \neq t^l_{2\xi+1}$ (as $\zeta \neq \xi$) so $t^l_{2\zeta+1} < t^l_{2\xi+1}$ as required.

Condition γ. If $l \notin w$ (but $l \in \{1, \ldots, n\}$) then $t^l_{2\zeta+1} > t^l_{2\xi+1}$.

Possibility γ1. $l \in u^*$.

Then $t^l_{2\zeta} = s_\zeta^{4l-1}$, $t^l_{2\xi} = s_\xi^{4l-1}$ (check the definition of the s's). Now by $(*)_5$(b):

$$x_\zeta \cap [p_l, r_{l+1}) = [s_\zeta^{4l-1}, s_\zeta^{4l}),$$

hence (by case 2 above)

$$x_\zeta \cap [p_l, r_{l+1}) = [t^l_{2\zeta+1}, q_\zeta^{l,4});$$

and

$$x_\xi \cap [p_l, r_{l+1}) = [s_\xi^{4l-1}, s_\xi^{4l}),$$

hence (by case 2 above)

$$x_\xi \cap [p_l, r_{l+1}) = [t^l_{2\xi+1}, q_\xi^{l,4}).$$

But as we are in Case A, $x_\zeta \subseteq x_\xi$ hence $x_\xi \cap [p_l, r_{\zeta+1}) \subseteq x_\xi \cap [p_l, r_{l+1})$, which means by the previous sentence $[t^l_{2\zeta+1}, q_\zeta^{l,4}) \subseteq [t^l_{2\xi+1}, 1_\xi^{l,4})$, which implies

$t^l_{2\zeta+1} \geqq t^l_{2\xi+1}$ and $q^{l,4}_\xi \geqq q^{l,4}_\zeta$. But $t^l_{2\zeta+1} \neq t^l_{2\xi+1}$ (as $\zeta \neq \xi$) so $t^l_{2\zeta+1} > t^l_{2\xi+1}$ as required.

Possibility γ2. $l \notin u^*$.

Then $t^l_{2\zeta+1} = s^{4l-3}_\zeta$, $t^l_{2\xi+1} = s^{4l-3}_\xi$ (check the definition of the s's); now by $(*)_5$(a):

$$x_\zeta \cap [r_l, p_l) = [s^{4l-3}_\zeta, s^{4l-2}_\zeta),$$

hence

$$x_\zeta \cap [r_l, p_l) = [t^l_{2\zeta+1}, q^{l,2}_\zeta);$$

and

$$x_\xi \cap [r_l, p_l) = [s^{4l-3}_\xi, s^{4l-2}_\xi),$$

hence

$$x_\xi \cap [r_l, p_l) = [t^l_{2\xi+1}, q^{l,2}_\xi).$$

But as we are in Case A, $x_\zeta \subseteq x_\xi$, hence $x_\zeta \cap [r_l, p_l) \subseteq x_\xi \cap [r_l, p_l)$, which means by the previous sentence $[t^l_{2\zeta+1}, q^{l,2}_\zeta) \subseteq [t^l_{2\xi+1}, q^{l,2}_\xi)$, which implies $t^l_{2\xi+1} \leqq t^l_{2\zeta+1}$ and $q^{l,2}_\zeta \leqq q^{l,2}_\xi$. But $t^l_{2\zeta+1} \neq t^l_{2\xi+1}$ (as $\zeta \neq \xi$) so $t^l_{2\xi+1} < t^l_{2\zeta+1}$ as required.

Case B. $x_\xi \subseteq x_\zeta$.

In this case we shall prove that 2ζ, 2ξ are a pair of ordinals we are looking for; i.e. conditions (α), (β), (γ) below hold and we shall check those, thus finishing this case (hence 4.8).

Condition α. $2\zeta < 2\xi$.

[Trivial by $\zeta < \xi$.]

Condition β. If $l \in w$ then $t^l_{2\zeta} < t^l_{2\xi}$.

Possibility β1. $l \in u^*$.

Then $t^l_{2\zeta} = s^{4l-3}_\zeta$, $t^l_{2\xi} = s^{4l-3}_\xi$ (check the definition of the s's); now by $(*)_4$(a):

$$x_\zeta \cap [r_l, p_l) = [s^{4l-3}_\zeta, s^{4l-2}_\zeta),$$

hence (by case 1 above)

$$x_\zeta \cap [r_l, p_l) = [t^l_{2\zeta}, q^{l,2}_\zeta);$$

and

$$x_\xi \cap [r_l, p_l) = [s^{4l-3}_\xi, s^{4l-2}_\xi),$$

hence (by case 1 above)

$$x_\xi \cap [r_l, p_l) = [t^l_{2\xi}, q^{l,2}_\zeta).$$

But as we are in Case B, $x_\zeta \supseteq x_\xi$ hence $x_\zeta \cap [r_l, p_l) \supseteq x_\xi \cap [r_l, p_l)$, which means by the previous sentence $[t^l_{2\zeta}, q^{l,2}_\zeta) \supseteq [t^l_{2\xi}, q^{l,2}_\xi)$, which implies $t^l_{2\zeta} \leqq t^l_{2\xi}$ and $q^{l,2}_\xi \leqq q^{l,2}_\zeta$. But $t^l_{2\zeta} \neq t^l_{2\xi}$ (as $\zeta \neq \xi$), so $t^l_{2\zeta} < t^l_{2\xi}$ as required.

Possibility β2. If $l \notin u^*$ (but $l \in \{1, \ldots, n\}$ then $t^l_{2\zeta} = s^{4l-1}_\zeta$, $t^l_{2\xi} = a^{4l-1}_\zeta$ (check the definition of the s's); now by $(*)_5$(b):

$$x_\xi \cap [p_l, r_{l+1}) = [s^{4l-1}_\zeta, s^{4l}_\zeta),$$

hence (by case 3 above)

$$x_\zeta \cap [p_l, r_{l+1}) = [t^l_{2\zeta}, q^{l,4}_\zeta);$$

and

$$x_\xi \cap [p_l, r_{l+1}) = [s^{4l-1}_\xi, s^{4l}_\xi),$$

hence (by case 3 above)

$$x_\xi \cap [p_l, r_{l+1}) = [t^l_{2\xi}, q^{l,4}_\xi).$$

But as we are in Case B, $x_\zeta \supseteq x_\xi$ hence $x_\zeta \cap [p_l, r_{l+1}) \supseteq x_\xi \cap [p_l, r_{l+1})$, which means by the previous sentence $[t^l_{2\zeta}, q^{l,4}_\zeta) \supseteq [t^l_{2\xi}, q^{l,4}_\xi)$, which implies $t^l_{2\zeta} \leqq t^l_{2\xi}$ and $q^{l,4}_\xi \leqq q^{l,4}_\zeta$. But $t^l_{2\zeta} \neq t^l_{2\xi}$ (as $\zeta \neq \xi$), so $t^l_{2\zeta} < t^l_{2\xi}$ as required.

Condition γ. $l \notin w$ (but $l \in \{1, \ldots, n\}$), then $t^l_{2\zeta} > t^l_{2\xi}$.

Possibility γ1. $l \in u^*$.
Then $t^l_{2\zeta} = s^{4l-2}_\zeta$, $t^l_{2\xi} = s^{4l-2}_\xi$ (check the definition of the s's); now by $(*)_5$(a):

$$x_\zeta \cap [r_l, p_l) = [s^{4l-3}_\zeta, s^{4l-2}_\zeta),$$

hence (by case 2 above)

$$x_\zeta \cap [r_l, p_l) = [q^{l,1}_\zeta, t^l_{2\zeta});$$

and

$$x_\xi \cap [r_l, p_l) = [s^{4l-2}_\xi, s^{4l-2}_\xi),$$

hence (by case 2 above)

$$x_\xi \cap [r_l, p_l) = [q^{l,1}_\xi, t^l_{2\xi}).$$

But as we are in Case B, $x_\zeta \supseteq x_\xi$ hence $x_\zeta \cap [r_l, p_l) \supseteq x_\xi \cap [r_l, p_l)$, which means by the previous sentence $[q^{l,1}_\zeta, t^l_{2\zeta}) \supseteq [q^{l,1}_\xi, t^l_{2\xi})$, which implies $q^{l,1}_\zeta \leqq q^{l,1}_\xi$ and $t^l_{2\xi} \leqq t^l_{2\zeta}$. But $t^l_{2\zeta} \neq t^l_{2\xi}$ (as $\zeta \neq \xi$), so $t^l_{2\zeta}$ as required.

Possiblity $\gamma 2$. $l \notin u^*$.

Then $t^l_{2\zeta} = s^{4l}_\zeta$, $t^l_{2\xi} = s^{4l}_\xi$ (check the definition of the s's); now by $(*)_5$(b):

$$x_\zeta \cap [p_l, r_{l+1}) = [s^{4l-1}_\zeta, s^{4l}_\zeta),$$

hence

$$x_\zeta \cap [p_l, r_{l+1}) = [q^{l,3}_\zeta, t^l_{2\zeta});$$

and (by case 4 above)

$$x_\xi \cap [p_l, r_{l+1}) = [s^{4l-1}_\xi, s^{4l}_\xi),$$

hence (by case 4 above)

$$x_\xi \cap [p_l, r_{l+1}) = [q^{l,3}_\xi, t^l_{2\xi}).$$

But as we are in Case B, $x_\zeta \supseteq x_\xi$ hence $x_\zeta \cap [p_l, r_{l+1}) \supseteq x_\xi \cap [p_l, r_{l+1})$, which means by the previous sentence $[q^{l,3}_\zeta, t^l_{2\zeta}) \supseteq [q^{l,3}_\xi, t^l_{2\xi})$, which implies $q^{l,3}_\zeta \leqq q^{l,3}_\xi$ and $t^l_{2\xi} \leqq t^l_{2\zeta}$. But $t^l_{2\zeta} \neq t^l_{2\xi}$ (as $\zeta \neq \xi$), so $t^l_{2\xi} < t^l_{2\zeta}$ as required.

So we finish the proof of 4.8.

4.9. Fact. (1) There is an entangled linear order $A \subseteq \mathbf{R}$ of power $\mathrm{cf}(2^{\aleph_0})$.

(2) Generalization to higher cardinals: if there is a linear order of power $\mathrm{cf}(2^\lambda)$ and density λ (e.g. λ strong limit), *then* there is an entangled linear order of power 2^λ and density λ.

Proof. Done independently by Bonnet-Shelah [BoSh 210], Todorcevic [T 1].

4.10. Fact. Suppose $\langle \lambda_i : i < \delta \rangle$ is a strictly increasing sequence of regular cardinals, $\bigwedge_{i<\delta} \lambda_i < \lambda = \mathrm{cf}\,\lambda$, $\lambda_i > |\delta|$, D a filter on δ, $\mathrm{cf}(\Pi_{i<\delta} \lambda_i / D) = \lambda$, i.e. there is $\langle f_\alpha : \alpha < \lambda \rangle \subset \Pi_{i<\delta} \lambda_i$ such that for every ultrafilter E extending D one has:

(i) $\alpha < \beta < \lambda \Rightarrow f_\alpha <_E f_\beta$,

(ii) $(\forall f \in \Pi_{i<\delta} \lambda_i)(\exists \alpha < \lambda)(f <_E f_\alpha)$.

Suppose $A_i \subseteq \delta$ $(i < \kappa)$ are such that, in $\mathscr{P}(\delta)/D$, $\{A_i : i < \kappa\}$ is independent and for $i < \delta$, $|\{f_\alpha \restriction i : \alpha < \lambda\}| < \lambda_i$. *Then* $\mathrm{Ens}(\lambda, \kappa)$.

4.10A. Remark. If $\mu > 2^{\mathrm{cf}\,\mu}$ then there are such $\langle \lambda_i : i < \delta \rangle$ and D for $\kappa = 2^{\mathrm{cf}\,\mu}$, $\lambda = \mu^+$ by §7.

Proof. Let $I = \{f_\alpha : \alpha < \lambda\}$. For each $\zeta < \kappa$ we define a linear order $<^*_\zeta$ of I:

$$f_\alpha <^*_\zeta f_\beta \qquad \textit{iff} \text{ for some } i < \delta:$$

$$f_\alpha(i) \neq f_\beta(i) \;\&\; f_\alpha \restriction i = f_\beta \restriction i \;\&\; [f_\alpha(i) < f_\beta(i) \Leftrightarrow i \in A_\zeta].$$

Let $n<\omega$, $\zeta_1<\cdots<\zeta_n<\kappa$. For $l=1,\ldots,n$, $t^l_\gamma=f_{\alpha(l,\gamma)}$ are pairwise distinct for $\gamma<\lambda$; and let $w\subseteq\{1,\ldots,n\}$. Let

$$g_\gamma(i)\overset{\text{def}}{=}\operatorname{Min}\{f_{\alpha(l,\gamma)}(i):l\in\{1,\ldots,n\}\},$$

$$i_\gamma\overset{\text{def}}{=}\operatorname{Min}\{i:\langle f_{\alpha(l,\gamma)}\restriction i;l\in\{1,\ldots,n\}\rangle\text{ are pairwise distinct}\}.$$

W.l.o.g. $i_\gamma=i^*$ for every γ.

Let $B=\{i<\delta$: for every $\xi<\lambda_i$, there are λ ordinals $\gamma<\lambda$ such that $g_\gamma(i)>\xi\}$.

CLAIM. $B\in D$.

PROOF. Suppose that $B\notin D$. Then, since $D=\bigcap\{F:F\supset D\,\&\,F$ is an ultrafilter on $\delta\}$, there is an ultrafilter F on δ, $B\notin F$. So $C:=\delta-B\in F$. From the definition of B,

$$(\forall i\in C)(\exists\xi_i<\lambda_i)(\exists\gamma_i<\lambda)(\gamma_i\leqq\gamma<\lambda\Rightarrow g_\gamma(i)\leqq\xi_i).$$

Define $h\in\Pi_{i<\delta}\lambda_i$ by

$$h(i):=\begin{cases}\xi_i+1 & \text{if } i\in C;\\ 0 & \text{if } i\notin C.\end{cases}$$

$\langle f_\alpha/D:\alpha<\lambda\rangle$ is cofinal in $\Pi_{i<\delta}\lambda_i/D$, hence $\langle f_\alpha/F:\alpha<\lambda\rangle$ is cofinal in $\Pi_{i<\delta}\lambda_i/F$, so there exists $\beta<\lambda$ such that

$$h<f_\beta \bmod F.$$

W.l.o.g. $\bigcup_{i\in C}\gamma_i<\beta$ [since $C\subseteq\delta$, $|\delta|<\lambda=\operatorname{cf}(\lambda)$ and $\bigwedge_{i\in C}(\gamma_i<\lambda)$]. Since $\alpha(l,\zeta)$, $(1\leqq l\leqq n,\ \zeta<\lambda)$ are pairwise distinct, and $\beta<\lambda$, there exists $\zeta<\lambda$ such that $\bigwedge_{l=1}^n(\alpha(l,\zeta)>\beta)$. W.l.o.g. $\bigcup_{i\in C}\gamma_i<\zeta$. So $\bigwedge_{l=1}^n(f_\beta<f_{\alpha(l,\zeta)}\bmod F)$. That means

$$E:=\left\{i<\delta:\bigwedge_{l=1}^{n}f_\beta(i)<f_{\alpha(l,\zeta)}(i)\right\}\in F.$$

So $E=\{i<\delta:f_\beta(i)<g_\zeta(i)\}\in F$, using the definition of g_ζ. Since $h<f_\beta \bmod F$, it now follows that $\{i<\delta:h(i)<g_\zeta(i)\}\in F$ and so $C\cap\{i<\delta:h(i)<g_\zeta(i)\}\in F$. Choosing i in this (non-empty) intersection, one obtains

$$g_\zeta(i)\leqq\xi_i<\xi_i+1=h(i)<g_\zeta(i),$$

a contradiction. So $B\in D$, proving the claim.

Then choose $i<\delta$ as follows. First note that since $|\{f_\alpha\restriction i:\alpha<\lambda\}|<\lambda_i$ for each $i < \delta$, and $\mathrm{cf}(\Pi_{i<\delta}\lambda_i/D) = \lambda$, D cannot contain any bounded subsets of δ. By hypothesis,

$$A := \bigcap_{l \in w} A_{\zeta_l} \cap \bigcap_{l \notin w} (\delta - A_{\zeta_l}) \notin D^*$$

(the dual ideal of D), so $\delta - A \notin D$ and there exists an ultrafilter F on δ such that $F \supset D$ and $A \in F$. Hence $C := \{i < \delta : i^* < i\} \cap A \cap B \in F$ and one can choose $i \in C$.

Then choose i:

$$i^* < i \in B \cap \bigcap_{l \in w} A_{\zeta_l} \cap \bigcap_{\substack{1 \leq l \leq n \\ l \notin w}} (\delta - A_{\zeta_l}).$$

For each $\xi < \lambda_i$ choose γ_ξ such that $g_{\gamma_\xi}(i) > \xi$. For some $S \subseteq \lambda_i$ unbounded $\xi_1 < \xi_2 \in S \Rightarrow \bigwedge_{l,m} f_{\alpha(l,\gamma_{\xi_1})}(i) < f_{\alpha(m,\gamma_{\xi_2})}(i)$. W.l.o.g. $\langle f_{\alpha(l,\gamma_\xi)} \restriction i : \xi \in S \rangle$ is constant (by a hypothesis). The conclusion should now be clear.

4.11. Fact. If $\langle \lambda_i : i < \delta \rangle$ is a strictly increasing sequence of regular cardinals $\bigwedge_{i<\delta} \lambda_i < \lambda = \mathrm{cf}\,\lambda$, $\lambda_i > |\delta|$, D an ultrafilter on δ, $\mathrm{cf}(\Pi_{i<\delta}\lambda_i/D) = \lambda$, and there is $\langle f_\alpha/D : \alpha < \lambda \rangle$ $<_D$-increasing cofinal in $\Pi_{i<\delta}\lambda_i/D$ such that for $i < \delta$ we have $\mu_i \stackrel{\mathrm{def}}{=} |\{f_\alpha \restriction i : \alpha < \lambda\}| < \lambda_i$ and $\mathrm{Ens}(\lambda_i, \mu_i)$, *then* there is an entangled linear order of power λ.

Proof. Let $\langle f_\alpha : \alpha < \lambda \rangle$ exemplify $\mathrm{cf}(\Pi\lambda_i/D) = \lambda$. Let $\langle I^i_\eta : \eta \in \Pi_i \rangle$ where $\Pi_i = \{f_\alpha \restriction i : \alpha < \lambda\}$ witness $\mathrm{Ens}(\lambda_i, \mu_i)$; w.l.o.g. I^i_η has universe λ_i.

Define $<^*$ on $I := \{f_\alpha : \alpha < \lambda\}$:

$$f_\alpha <^* f_\beta \qquad \textit{iff} \text{ there is } i < \delta \text{ such that:}$$

$$f_\alpha \restriction i = f_\beta \restriction i,$$

$$I^i_{f_\alpha \restriction i} \models f_\alpha(i) < f_\beta(i).$$

Checking — easy, choosing $i \in \{i < \delta : i^* < i\} \cap B$ and $S \subset \lambda_i$ in the notation of the proof of 4.10.

4.11A. Remark. So we have another way to get: if $\lambda = \beth_\lambda > \mathrm{cf}\,\lambda$, *then* for some regular $\kappa \in (\lambda, 2^\lambda)$ there is an entangled order.

4.12. Fact. Suppose $\langle \lambda_i : i < \delta \rangle$ is strictly increasing, D the filter of cobounded subsets of δ, $\mathrm{tcf}(\Pi\lambda_i/D)=\lambda$, $\mu<\mathrm{cf}\,\delta$, $\delta<\lambda_0$, $\mu<\lambda_0<\bigcup_{i<\delta}\lambda_i<\mathrm{Ded}\,\mu$, $2^\mu<\lambda$. *Then* $\mathrm{Ens}_2(\mathrm{cf}(\delta),\lambda)$.

PROOF. Let J be a dense linear order of power $\bigcup_{i<\delta}\lambda_i$ with a dense subset I of power μ. Let t^i_ζ ($i<\delta$, $\zeta<\lambda_i$) be distinct members of J. Let $\langle f_\alpha;\alpha<\lambda\rangle$ witness $\mathrm{tcf}(\Pi_{i<\delta}\lambda_i/D)=\lambda$. For each α let $I_\alpha=\{t^i_{f_\alpha(i)}:i<\delta\}$. For $\alpha<\lambda$ let $A_\alpha=\{\beta:I_\alpha,I_\beta$ are not $\mathrm{cf}(\delta)$-far$\}$. Now for each $\beta\in A_\alpha$ there are $K_{\alpha,\beta}\subseteq I_\alpha$, $L_{\alpha,\beta}\subseteq I_\beta$ each of power $\mathrm{cf}(\delta)$ and $h_{\alpha,\beta}$ an isomorphism or anti-isomorphism from $K_{\alpha,\beta}$ onto $L_{\alpha,\beta}$; let $M_{\alpha,\beta}$ be a dense subset of $K_{\alpha,\beta}$ of power $\leqq\mu$.† Assume $|A_\alpha|=\lambda$. As $2^\mu<\lambda$ for some $A'_\alpha\subseteq A_\alpha$, $|A'_\alpha|=\lambda$ and for some M^*_α, h_α we have: $[\beta\in A'_\alpha\Rightarrow M_{\alpha,\beta}=M^*_\alpha\ \&\ h_{\alpha,\beta}\restriction M^*_\alpha=h_\alpha]$. Essentially h_α defines uniquely $h_{\alpha,\beta}(x)$ where $x\in\mathrm{Dom}\,h_{\alpha,\beta}$. More fully, let

$$I^\alpha\stackrel{\mathrm{def}}{=}\{x\in I_\alpha:\text{there is }y\in J,\ x,y\text{ is single in the Dedekind cut it realizes in }M^*_\alpha,\ h''_\alpha(M^*_\alpha)\text{ respectively }(\forall z\in M^*_\alpha)[z<y\equiv h_\alpha(z)<x]\}.$$

Now $[\beta\in A'_\alpha\Rightarrow\mathrm{Dom}\,h_{\alpha,\beta}\subseteq I^\alpha\subseteq I_\alpha]$ and $h^\alpha\stackrel{\mathrm{def}}{=}\bigcup_{\beta\in A'_\alpha}h_{\alpha,\beta}$ is a function from I^α to J.

Now define $g^\alpha\in\Pi_{i<\delta}\lambda_i: g^\alpha(i)=\sup\{\zeta<\lambda_i:t^i_\zeta\in\mathrm{Rang}(h^\alpha)\}$, $g^\alpha(i)<\lambda_i$ as $|\mathrm{Rang}\,h^\alpha|=|\mathrm{Dom}\,h^\alpha|=|I^\alpha|\leqq|I_\alpha|\leqq|\delta|$ so $[\beta\in A'_\alpha\Rightarrow f_\beta\leqq g^\alpha]$. But $|A'_\beta|=\lambda$; contradiction. Hence $|A_\alpha|<\lambda$, so we can find an unbounded $A^*\subseteq\lambda$ such that

$$\alpha<\beta\wedge\alpha\in A^*\wedge\beta\in A^*\Rightarrow\beta\notin A_\alpha.$$

I.e. we have λ linear orders, each of power $\mathrm{cf}(\delta)>\mu$, any two are $\mathrm{cf}(\delta)$-far. By 4.7(2) we finish.

4.13. CLAIM. In Claim 4.10 suppose in addition μ is a limit cardinal, $\Pi_{i<\delta}\lambda_i\geqq\mu\geqq\mathrm{cf}\,\mu=\lambda$. Then

(1) $\mathrm{Ens}(\mu,\kappa)$.

(2) Moreover, there are $\langle I_\zeta:1+\zeta<\kappa\rangle$ exemplifying $\mathrm{Ens}(\mu,\kappa)$ such that:

  a) for each $\theta<\mu$ there is a linear order of power θ embeddable in every I_ζ;

  b) each I_ζ has dense subsets of power $\Sigma_{i<\delta}\lambda_i<\mu$.

PROOF. (1) Let $\mu=\bigcup_{\alpha<\lambda}\mu_\alpha$, $\mu_\alpha<\mu$, $[\alpha<\beta\Rightarrow\mu_\alpha<\mu_\beta]$ and $\langle f_\alpha/D:\alpha<\lambda\rangle$

† Such that if $x\in I$, $\mathrm{Min}\{y\in K_{\alpha,\beta}:y>x\}$ is well defined, then it is in $M_{\alpha,\beta}$; similarly with $\mathrm{Max}\{y\in K_{\alpha,\beta}:y<x\}$; similarly $h''_{\alpha,\beta}(M_{\alpha,\beta})$, $L_{\alpha,\beta}$.

be cofinal in $\Pi\lambda_i/D$. So for each α, as $\Pi_{i<\delta}\{\zeta : f_\alpha(i) < \zeta < \lambda_i\}$ has power $\Pi_{i<\delta}\lambda_i \geqq \mu$, it has a subset F_α of cardinality μ_α^+; as $\langle f_\alpha/D : \alpha < \lambda\rangle$ is cofinal in $\Pi_{i<\delta}\lambda_i/D$, for some $\gamma_\alpha < \lambda$,

$$F'_\alpha \stackrel{\text{def}}{=} \{g \in F_\alpha : g/D < f_{\gamma_\alpha}/D\} \text{ has power } \geqq \mu_\alpha$$

(and w.l.o.g. $\gamma_\alpha = \alpha + 1$). Let $I = \bigcup_{\alpha<\lambda} F'_\alpha$ and proceed as before (in 4.10).
(2) W.l.o.g. $A \stackrel{\text{def}}{=} \bigcap_{\zeta<\kappa} A_\zeta$ is such that $\Pi_{i\in A}\lambda_i \geqq \mu$. [Why? Let us use $\langle A'_\zeta : \zeta < \kappa\rangle$ where $A'_\zeta \stackrel{\text{def}}{=} A_0 \cup A_{1+\zeta}$ if $\Pi_{i\in A_0}\lambda_i \geqq \mu$ and $A'_\zeta = (\delta - A_0) \cup A_{1+\zeta}$ if $\Pi_{i\in A_0}\lambda_i < \mu$.] Now we can choose $F_\alpha \subseteq \Pi\lambda_i$ such that:

(i) $|F_\alpha| = \mu_\alpha$,
(ii) for some $\gamma_\alpha < \lambda$, $g \in F_\alpha \Rightarrow f_\alpha \leqq g \leqq_D f_{\gamma_\alpha}$,
(iii) $g, h \in F_\alpha \Rightarrow g \restriction (\delta - A) = h \restriction (\delta - A)$.

So on F_α all orders $<^*_\zeta$ are the same, and so $\langle(\bigcup_{\alpha<\lambda} F_\alpha, <^*_\zeta) : \zeta < \kappa\rangle$ are as required.

4.14. Theorem. *If the conclusion of* 4.13(2) *holds for* $\kappa = 3$ (*i.e. pair of orders*), then *for some Boolean algebra* B *the spread of* B *is* μ *but it is neither obtained nor weakly obtained.*

Proof. By Todorcevic's proof of [M] 1.9 from [M] 1.4 in Monk [M] (also the part on: "$s(B/K)$ is obtained for every ideal K of B" generalized; but see 4.3).

4.15. Conclusion. If $\theta = \mathrm{cf}\,\lambda$, $(\forall\chi < \lambda)[\chi^\theta < \lambda]$, θ uncountable (or at least $\sup\{\mathrm{cf}\,\Pi_{i<\theta}\lambda_i : \lambda_i < \lambda\}$ is λ^θ or just $\geqq \mathrm{cf}\,\mu$), *then*:

(a) for every μ, $\lambda < \mathrm{cf}\,\mu \leqq \mu \leqq \lambda^\theta$, $\mathrm{Ens}(\mu, 2^\theta)$;
(b) moreover this is exemplified by $\langle I_\zeta ; \zeta < 2^\theta\rangle$ where every I_ζ has density λ and for $\sigma < \mu$ there is an order of power σ embeddable into every I_ζ;
(c) for every limit cardinal μ, $\lambda < \mathrm{cf}\,\mu \leqq \mu \leqq \lambda^\theta$ for some Boolean algebra A, $s(A) = \mu$ but it is not obtained (nor weakly obtained).

4.15A. Remark. We shall return to this in light of the additional information on cofinalities of products of regular cardinals. I.e. if $\mu = \chi^+$, $\mathrm{cf}\,\chi = \theta < \chi$, the conclusion holds.

Proof. By 9.3, letting D be the cobounded filter on θ and $A^*_i \subseteq \theta$ pairwise disjoint for $i < \theta$, $A^*_i \neq \emptyset \bmod D$ *there is* $\langle\lambda_i : i < \theta\rangle$ a strictly increasing sequence of regular cardinals $< \lambda$ such that $\Pi_{i<\theta}\lambda_i/D$ has cofinality $\mathrm{cf}\,\mu$; so w.l.o.g. $\lambda_i > \Pi_{j<i}\lambda_j$. Let $\langle w_i : i < 2^\theta\rangle$ be independent in $\mathscr{P}(\theta)$. Let

$A_i = \bigcup_{j \in w_i} A_j^*$. Now D, $\langle A_i : i < 2^\theta \rangle$, $\langle \lambda_i : i < \theta \rangle$ are as required in 4.13 and we get the conclusions by 4.14.

4.16. Fact. In 4.12, suppose in addition $\mathrm{cf}\,\chi = \mathrm{cf}\,\delta < \chi \leqq \bigcup_{i<\delta} \lambda_i$. Then we can find $\langle I_\zeta : \zeta < \lambda \rangle$ such that:

(a) I_ζ is a linear order of power χ with a dense subset of power μ;
(b) the linear orders $\{I_\zeta ; \zeta < \lambda\}$ are pairwise far.

Proof. Use 4.12, $D = \{A \subset S : \delta - A$ is bounded$\}$, $\chi = \Sigma_{i<\delta} \chi_i$, $\chi_i > \Sigma_{j<i} \chi_j$; replace t_ζ^i by χ_i elements.

§5. The basic properties of pcf(a)

Notation. Let a, b, c denote sets of regular cardinals. J denotes an ideal (usually on some a), D a filter. For a set A of ordinals with no last element, $J_A^{\mathrm{bd}} = \{B \subseteq A ; \sup B < \sup A\}$, i.e. the ideal of bounded subsets.

5.1. Definition. (1) For a partial order P:

(a) P is λ-directed if, for every $A \subseteq P$, $|A| < \lambda$, there is $q \in P$ such that $\bigwedge_{p \in A} p \leqq q$ (q is an upper bound of A);
(b) P has true cofinality λ *if* there is $\langle p_i : i < \lambda \rangle$ cofinal in P, i.e.

$$\bigwedge_{i<j} p_i < p_j, \qquad \forall q \in P \left[\bigvee_i q \leqq p_i \right]$$

[and one writes $\mathrm{tcf}(P) = \lambda$ for the minimal λ]
(if P is linearly ordered it always has a true cofinality);
(c) P is endless if $\forall p \in P\ \exists q \in P [p < q]$ (so if P is endless, in (a), (b), (d) we can replace $\leqq$ by $<$);
(d) $A \subseteq P$ is a cover if $\forall p \in P\ \exists q \in A [p \leqq q]$;
(e) $\mathrm{cf}(P) = \mathrm{Min}\{|A| : A \subseteq P$ is a cover$\}$.

(2) $R^{\kappa,1} = \{\lambda : \lambda = \mathrm{cf}\,\lambda > \kappa\}$.

(3) If D is a filter on S, α_s (for $s \in S$) are ordinals, $f, g \in \Pi_{s \in S} \alpha_s$, then $f/D < g/D$, $f <_D g$ and $f < g \bmod D$ all mean $\{s \in S : f(s) < g(s)\} \in D$. Similarly for $\leqq$, and we do not distinguish between a filter and the dual ideal in such notions. So if J is an ideal on a and $f, g \in \Pi a$,, then $f < g \bmod J$ iff $\{\theta \in a : \neg\, f(\theta) < g(\theta)\} \in J$.

(4) For $f, g : S \to$ Ordinals, $f < g$ means $\bigwedge_{s \in S} f(s) < g(s)$; similarly $f \leqq g$.

5.2. Definition. (1) For a property Γ of ultrafilters (if Γ is the empty condition, we omit it):

$$\mathrm{pcf}_\Gamma(a) = \{\mathrm{tcf}(\Pi a/D) : D \text{ is an ultrafilter on } a \text{ satisfying } \Gamma\}$$

(so it is a set of regular cardinals).

(2) $J^0_{<\lambda}[a] = \{b \subseteq a$: for no ultrafilter D on a to which b belongs is $\mathrm{tcf}(\Pi a/D) \geqq \lambda\}$.

5.3. Claim. (0) $(\Pi a, <_J)$, $(\Pi a \leqq_J)$ are endless.

(1) $\mathrm{Min}(\mathrm{pcf}(a)) \geqq \mathrm{Min}\, a$.

(2) If $a \subseteq b$ then $\mathrm{pcf}(a) \subseteq \mathrm{pcf}(b)$; and for any b, c $\mathrm{pcf}(c \cup b) = \mathrm{pcf}(c) \cup \mathrm{pcf}(b)$ and:

$$x \in J^0_{<\lambda}[b \cup c] \Leftrightarrow x \subseteq c \cup b \wedge x \cap c \in J^0_{<\lambda} \quad [c] \wedge x \cap b \in J^0_{<\lambda}[b].$$

(3) (i) if $b \subseteq a, b$ finite, then $\mathrm{pcf}(b) = b$ and $\mathrm{pcf}(a) - b \subseteq \mathrm{pcf}(a - b) \subseteq \mathrm{pcf}(a)$;
(ii) in addition if $b \subseteq \{\theta \in a : |\theta \cap a| < \aleph_0\}$, then $\mathrm{pcf}(a - b) = \mathrm{pcf}(a) - b$; e.g. $b = \{\mathrm{Min}(a)\}$;
(iii) in addition if $\lambda > \max b$, and $\langle \Pi(a - b), <_{J^0_{<\lambda}[a-b]}\rangle$ is λ-directed, then $\langle \Pi a, <_{J^0_{<\lambda}[a]}\rangle$ is λ-directed.

(4) If D is an ultrafilter on a such that, for every $\theta \in a$, $(a - \theta^+) \in D$, *then* $\mathrm{cf}(\Pi a/D) \geqq \sup a$ (and if equality holds, then $\sup a$ is an inaccessible cardinal, D a weakly normal ultrafilter).

(5) If a has no last element, *then* there is $\lambda \in \mathrm{pcf}(a)$ such that $\sup a < \lambda$.

(6) If D is an ultrafilter on a set S and for $s \in S$, α_s is a limit ordinal *then* $\mathrm{cf}(\Pi_{s \in S}\, \alpha_s, <_D) = \mathrm{cf}(\Pi_{s \in S}\, \mathrm{cf}\, \alpha_s, <_D) = \mathrm{cf}(\Pi_{s \in S}(\alpha_s, <)/D)$, and

$$\mathrm{tcf}\left(\prod_{s \in S} \alpha_s, <_D\right) = \mathrm{tcf}\left(\prod_{s \in S} \mathrm{cf}\, \alpha_s, <_D\right) = \mathrm{tcf}\left(\prod_{s \in S} (\alpha_s, <)/D\right).$$

(7) If D is an ultrafilter on a set S, λ_s a regular cardinal, then $\theta \overset{\mathrm{def}}{=} \mathrm{tcf}(\Pi \lambda_s, <_D)$ is well defined and $|S| < \mathrm{Min}\{\lambda_s : s \in S\}$ implies $\theta \in \mathrm{pcf}\{\lambda_s : s \in S\}$.

(8) If $|\mathrm{pcf}(a)| < \mathrm{Min}(a)$, then $\mathrm{pcf}(a)$ has a maximal element.

(9) If $|\mathrm{pcf}(a)| < \mathrm{Min}(a)$, then $\mathrm{pcf}(\mathrm{pcf}(a)) = \mathrm{pcf}(a)$; more generally, if $c \subseteq \mathrm{pcf}(a)$, $|a| < \mathrm{Min}(a)$, $|c| < \mathrm{Min}(a)$, *then* $\mathrm{pcf}(c) \subseteq \mathrm{pcf}(a)$.

(10) If there is no maximal element in $\mathrm{pcf}(a)$, then $\mathrm{cf}[\mathrm{otp}(\mathrm{pcf}(a))] > \mathrm{Min}(a)$; moreover, $\sup \mathrm{pcf}(a)$ is a (weakly) inaccessible cardinal.

Proof. E.g.

(8) Let $b \overset{\mathrm{def}}{=} \mathrm{pcf}(a)$ and assume b has no last element; then by 5.3(5) there is

$\lambda \in \text{pcf}(b)$, $\lambda > \sup(b)$. However, by 5.3(9), $b = \text{pcf}(a) = \text{pcf}(\text{pcf}(a)) = \text{pcf}(b)$; hence $\lambda \in b$ — contradiction.

(9) See 5.10.

(10) See 5.11.

5.4. Claim. (1) $J^0_{<\lambda}[a]$ is an ideal (of $\mathscr{P}(a)$).

(2) If $\lambda \leqq \mu$, then $J^0_{<\lambda}[a] \subseteq J^0_{<\mu}[a]$.

(3) If λ is singular, $J^0_{<\lambda}[a] = J^0_{<\lambda^+}[a]$.

(4) If $\lambda \notin \text{pcf}(a)$, then $J^0_{<\lambda}[a] = J^0_{<\lambda^+}[a]$.

5.5. Lemma. *If* $\text{Min}(a) \geqq |a|$, λ *a cardinal* $> |a|^+$, then $(\Pi a, <_{J^0_{<\lambda}[a]})$ *is* λ*-directed.*

Proof. By 5.3(3)(iii) w.l.o.g. $|a|, |a|^+ \notin a$ so $\text{Min}\, a > |a|^+$. Note: if $f \in \Pi a, f < f + 1 \in \Pi a$ (i.e. $(\Pi a, <_{J^0_{<\lambda}[a]})$ is endless). Let $F \subseteq \Pi a$, $|F| < \lambda$, and we shall prove that for some $g \in \Pi a$, $(\forall f \in F)(f \leqq g \bmod J^0_{<\lambda}[a])$. The proof is by induction on $|F|$. If $|F|$ is finite, this is trivial. Also if $|F| < \text{Min}\, a$ it is easy: let $g \in \Pi a$ be $g(\theta) = \sup\{f(\theta) : f \in F\}$. So assume $|F| = \mu$, $\text{Min}\, a \leqq \mu < \lambda$, so let $F = \{f^0_i : i < \mu\}$. By the induction hypothesis we can choose by induction on $i < \mu, f^1_i \in \Pi a$, such that:

(a) $f^0_i \leqq f^1_i \bmod J^0_{<\lambda}[a]$,

(b) for $j < i, f^1_j \leqq f^1_i \bmod J^0_{<\lambda}[a]$.

If μ is singular, there is $C \subseteq \mu$ unbounded, $|C| = \text{cf}\,\mu < \mu$, and by the induction hypothesis there is $g \in \Pi a$ such that for $i \in C, f^1_i \leqq g \bmod J^0_{<\lambda}[a]$. Now g is as required:

$$f^0_i \leqq f^1_i \leqq f^1_{\text{Min}(C - i)} \leqq g \bmod J^0_{<\lambda}[a].$$

So w.l.o.g. μ is regular, Now we define by induction on $\alpha < |a|^+, g_\alpha, i_\alpha = i(\alpha)$, $\langle b^\alpha_i : i < \mu \rangle$ such that:

(i) $g_\alpha \in \Pi a$,

(ii) for $\beta < \alpha$, $g_\beta \leqq g_\alpha$,

(iii) for $i < \mu$ let $b^\alpha_i \stackrel{\text{def}}{=} \{\theta \in a : f^1_i(\theta) > g_\alpha(\theta)\}$,

(iv) for each α, for every $i \in [i_\alpha, \mu)$, $b^\alpha_i \neq b^{\alpha+1}_i$ (and $i(\alpha) < \mu$).

We cannot carry this definition: by letting $i(*) = \sup\{i_\alpha : \alpha < |a|^+\}$, then $i(*) < \mu$ since $\mu = \text{cf}\,\mu$, $\mu \geqq \text{Min}\, a > |a|^+$.

We know that $b^\alpha_{i(*)} \neq b^{\alpha+1}_{i(*)}$ for $\alpha < |a|^+$ (by (iv)) and $b^\alpha_{i(*)} \subseteq a$ (by (iii)) and $[\alpha < \beta \Rightarrow b^\beta_{i(*)} \subset b^\alpha_{i(*)}]$ (by (ii)), together a contradiction.

Now for $\alpha = 0$ let g_α be f^1_0.

For α limit let $g_\alpha(\theta) = \bigcup_{\beta<\alpha} g_\beta(\theta)$ (note: $g_\alpha \in \Pi a$ as $\alpha < |a|^+ < \operatorname{Min} a$ and a is a set of regular cardinals).

For $\alpha = \beta + 1$, suppose that $\langle b_i^\beta : i < \mu \rangle$ is defined. If $b_i^\beta \in J^0_{<\lambda}[a]$ for unboundedly many $i < \mu$, then g_β is an upper bound for F and the proof is complete. So assume this fails; then there is a bounding $i(\beta) < \mu$ such that $b^\beta_{i(\beta)} \notin J^0_{<\lambda}[a]$. As $b^\beta_{i(\beta)} \notin J^0_{<\lambda}[a]$, for some ultrafilter D on a, $b^\beta_{i(\beta)} \in D$ and $\operatorname{cf}(\Pi a/D) \geqq \lambda$. Hence $\{f_i^1/D : i < \mu\}$ has a bound h_α/D, $h_\alpha \in \Pi a$. Let us define $g_\alpha \in \Pi a$:

$$g_\alpha(\theta) = \operatorname{Max}\{g_\beta(\theta), h_\alpha(\theta)\}.$$

Now (i), (ii) hold trivially and b_i^α is defined by (iii). Why does (iv) hold with $i_\alpha := i(\beta)$? Suppose $i(\beta) \leqq i < \mu$. As $f^1_{i(\beta)} \leqq f_i^1 \bmod J^0_{<\lambda}[a]$ clearly $b^\beta_{i(\beta)} \subseteq b_i^\beta \bmod J^0_{<\lambda}[a]$. Moreover $J^0_{<\lambda}[a]$ is disjoint to D (by its definition) so $b^\beta_{i(\beta)} \in D$ implies $b_i^\beta \in D$.

On the other hand, b_i^α is $\{\theta \in a : f_i^1(\theta) > g_\alpha(\theta)\}$ which is equal to $\{\theta \in a : f_i^1(\theta) > g_\beta(\theta), h_\alpha(\theta)\}$, which does not belong to D (h_α was chosen such that $f_i^1 \leqq h_\alpha \bmod D$). We can conclude $b_i^\alpha \notin D$, whereas $b_i^\beta \in D$; so they are distinct.

Now we have said that we cannot carry the definition for all $\alpha < |a|^+$, so we are stuck at some α; by the above α is successor, say $\alpha = \beta + 1$, and g_β as required to bound F.

5.6. Lemma. *If* $\operatorname{Min} a \geqq |a|$, *$D$ is an ultrafilter on a and* $\lambda = \operatorname{tcf}(\Pi a, <_D)$, *then for some* $b \in D$, $(\Pi b, <_{J^0_{<\lambda}[a]})$ *has true cofinality* λ. (So $b \in J^0_{<\lambda^+}[a] - J^0_{<\lambda}[a]$.)

Proof. Again w.l.o.g. $\operatorname{Min} a > |a|^+$; and we know $\lambda \geqq \operatorname{Min} a$. Let $\langle f_i/D : i < \lambda \rangle$ be increasing unbounded in $\Pi a/D$ (so $f_i \in \Pi a$). By 5.5 w.l.o.g. $(\forall j < i)(f_j < f_i \bmod J^0_{<\lambda}[a])$. Now 5.6 follows from

5.7. Lemma. *Suppose* $|a| < \operatorname{Min}(a)$, $f_i \in \Pi a$, $f_i < f_j \bmod J^0_{<\lambda}[a]$ *for* $i < j < \lambda$, *and there is no* $g \in \Pi a$ *such that for every* $i < \lambda$, $f_i < g \bmod J^0_{<\lambda}[a]$.

Then there are b_i $(i < \lambda)$ *such that*:

(A) $b_i \subseteq a$, $b_i \notin J^0_{<\lambda}[a]$,

(B) $i < j \Rightarrow b_i \subseteq b_j \bmod J^0_{<\lambda}[a]$ (*i.e.* $b_i - b_j \in J^0_{<\lambda}[a]$),

(C) *for each* i, $\langle f_j \restriction b_i : j < \lambda \rangle$ *is cofinal in* $(\Pi b_i, <_{J^0_{<\lambda}[a]})$,

(D) *for some* $g \in \Pi a$, $\bigwedge_{i<\lambda} f_i < g \bmod J$ *where* $J = J^0_{<\lambda}[a] + \{b_i : i < \lambda\}$; *in fact*

(D)$^+$ *for some* $i(*) < \lambda$, $f_{i(*)+i} < g \bmod (J^0_{<\lambda}[a] + b_i)$,

(E) *if* $g \leqq g' \in \Pi a$, *then for arbitrarily large* $i < \lambda$

$$\bigwedge_{\theta\in a} [g(\theta)\geqq f_i(\theta)\Leftrightarrow g'(\theta)\geqq f_i(\theta)].$$

Proof of 5.7. Assume the lemma fails. We now define by induction on $\alpha<|a|^+$, g_α, $i(\alpha)$, $\langle b_i^\alpha : i<\lambda\rangle$ such that:

(i) $g_\alpha\in\Pi a$,

(ii) for $\beta<\alpha$, $g_\beta\leqq g_\alpha$,

(iii) $b_i^\alpha\overset{\text{def}}{=}\{\theta\in a: f_i(\theta)>g_\alpha(\theta)\}$,

(iv) if $i(\alpha)\leqq i<\lambda$ then $b_i^\alpha\neq b_i^{\alpha+1}$.

For $\alpha=0$ let $g_\alpha=f_0$.

For α limit let $g_\alpha(\theta)=\bigcup_{\beta<\alpha} g_\beta(\theta)$ (now $[\beta<\alpha\Rightarrow g_\beta\leqq g_\alpha]$ trivially and $g_\alpha\in\Pi a$ as $\mathrm{Min}\, a\geqq|a|^+>\alpha$).

For $\alpha=\beta+1$, if $\{i<\lambda: b_i^\beta\in J^0_{<\lambda}[a]\}$ is unbounded in λ, then g_β is a bound for $\langle f_i : i<\lambda\rangle \bmod J^0_{<\lambda}[a]$. So let $i(\beta)$ be such that $\forall i\in[i(\beta),\lambda)$, $b_i^\beta\notin J^0_{<\lambda}[a]$. If $\langle b_i^\beta : i(\beta)\leqq i<\lambda\rangle$ satisfies the desired conclusion we are done.

Now among the conditions in the conclusion of 5.7, (A) holds by assumption, (B) holds by b_i^β's definition as $[i<j\Rightarrow f_i<f_j \bmod J^0_{<\lambda}[a]]$, (D)$^+$ holds with $g=g_\beta$ by the choice of b_i^β. Lastly if (E) fails, say for g', then it can serve as g_α. So only (C) (of 5.7) may fail, w.l.o.g. for $i=i(\beta)$. I.e. $\langle f_j\restriction b^\beta_{i(\beta)} : j<\lambda\rangle$ is not cofinal in $(\Pi b^\beta_{i(\beta)}, <_{J^0_{<\lambda}[a]})$. As this sequence of functions is increasing w.r.t. $<_{J^0_{<\lambda}[a]}$, there is $h_\alpha\in\Pi b^\beta_{i(\beta)}$ such that for no $j<\lambda$, $h_\alpha\leqq f_j\restriction b^\beta_{i(\beta)} \bmod J^0_{<\lambda}[a]$. Let $h'_\alpha=h_\alpha\cup 0_{(a-b^\beta_{i(\beta)})}$, and $g_\alpha\in\Pi a$ be defined by $g_\alpha(\theta)=\mathrm{Max}\{g_\beta(\theta), h'_\alpha(\beta)\}$. Now define b_i^α by (iii) so (i), (ii), (iii) hold trivially, and we have to check (iv). So we can define g_α, $i(\alpha)$ for $\alpha<|a|^+$, satisfying (i)–(iv). As in the proof of 5.5, this is impossible; so that lemma cannot fail.

5.8. Lemma. *Suppose* $|a|<\mathrm{Min}(a)$.

(1) *For every* $b\in J^0_{<\lambda^+}[a]-J^0_{<\lambda}[a]$, *we have*: $(\Pi b, <_{J^0_{<\lambda}[a]})$ *has true cofinality* λ.

(2) *If* $0<\alpha<\lambda$ *and for* $\beta<\alpha$, $c_\beta\in J^0_{<\lambda^+}[a]-J^0_{<\lambda}[a]$, then *for some* $c\in J^0_{<\lambda^+}[a]-J^0_{<\lambda}[a]$:

$$\textit{for each } \beta<\alpha,\qquad c_\beta\subseteq c \bmod J^0_{<\lambda}[a].$$

(3) *If D is an ultrafilter on a, then* $\mathrm{cf}(\Pi a/D)$ *is* $\mathrm{Min}\{\lambda : D\cap J^0_{<\lambda^+}[a]\neq\varnothing\}$.

(4) *For λ limit*, $J^0_{<\lambda}[a]=\bigcup_{\mu<\lambda} J^0_{<\lambda}[a]$.

(5) $|\mathrm{pcf}(a)|\leqq 2^{|a|}$ *and* $[\lambda\in\mathrm{pcf}(a)\Leftrightarrow J^0_{<\lambda}[a]\neq J^0_{<\lambda^+}[a]]$.

Proof. (1) Let

$J = \{b \subseteq a : b \in J^0_{<\lambda}[a]$ or $b \in J^0_{<\lambda^+}[a] - J^0_{<\lambda}[a]$ and $(\Pi b, <_{J^0_{<\lambda}[a]})$ has true cofinality $\lambda\}$.

Clearly $J \subseteq J^0_{<\lambda^+}[a]$; it is quite easy to check it is an ideal. Assume $J \neq J^0_{<\lambda^+}[a]$ and we shall get a contradiction. Choose $b \in J^0_{<\lambda^+}[a] - J$; as J is an ideal, there is an ultrafilter D on a such that $D \cap J = \emptyset$ and $b \in D$. Now if $\mathrm{cf}(\Pi a/D) \geqq \lambda^+$, then $b \notin J^0_{<\lambda^+}[a]$ (by the definition of $J^0_{<\lambda^+}[a]$); contradiction. On the other hand, if $F \subseteq \Pi a$, $|F| < \lambda$, there is $g \in \Pi a$ such that $(\forall f \in F)(f < g \bmod J^0_{<\lambda}[a])$ (by 5.5), so $(\forall f \in F)[f < g \bmod D]$ (as $J^0_{<\lambda}[a] \subseteq J$, $D \cap J = \emptyset$), and this says $\mathrm{cf}(\Pi a/D) \geqq \lambda$. By the last two sentences we know that $\mathrm{cf}(\Pi a/D)$ is λ. Now by 5.6 for some $c \in D$, $(\Pi c, <_{J^0_{<\lambda}[a]})$ has true cofinality λ. Clearly if $c' \subseteq c$, $c' \notin J^0_{<\lambda}[a]$, then also $(\Pi c', <_{J^0_{<\lambda}[a]})$ has cofinality λ, hence w.l.o.g. $c \subseteq b$; hence $c \in J^0_{<\lambda^+}[a]$, hence by the definition of J, $c \in J$. But this contradicts the choice of D as disjoint from J.

We have to conclude that $J = J^0_{<\lambda^+}[a]$ so we have proved 5.8(1).

(2) For each $\beta < \alpha$ let $\langle f^\beta_j : j < \lambda \rangle$ exemplify that $(\Pi a, <_{J^0_{<\lambda}[a]+(a-c_\beta)})$ has true cofinality λ; so $f^\beta_j \in \Pi a$ and

$$[j(1) < j(2) < \lambda \Rightarrow f^\beta_{j(2)} < f^\beta_{j(2)} \bmod((J^0_{<\lambda}[a]) + (a - c_\beta))]$$

and

$$((\forall g \in \Pi a)(\exists j < \lambda)[g < f^\beta_j \bmod([J^0_{<\lambda}[a]) + (a - c_\beta))]]).$$

By 5.5 we can define $f^*_j \in \Pi a$ by induction on $j < \lambda$ such that

(i) for $i < j$, $f^*_i < f^*_j \bmod J^0_{<\lambda}[a]$,

(ii) for each $\beta < \alpha$, $f^\beta_j \leqq f^*_j \bmod J^0_{<\lambda}[a]$.

Let $\langle b_i : i < \lambda \rangle$ be as guaranteed by 5.7 (for $\langle f^*_j : j < \lambda \rangle$). Clearly for each $\beta < \alpha$, $\langle f^*_j : j < \lambda \rangle$ is $<_{J^0_{<\lambda}[a]+(a-c_\beta)}$-increasing and cofinal. So for each $\beta < \alpha$ for some $i(\beta) < \lambda$

$$c_\beta \subseteq b_{i(\beta)} \bmod J^0_{<\lambda}[a].$$

[For if there is $\beta < \alpha$ such that $\neg (\bigvee_{i<\lambda} c_\beta \subseteq b_i \bmod J^0_{<\lambda}[a])$, then $c_\beta \notin J$, where J comes from 5.7(D). Choose now an ultrafilter D on a such that $c_\beta \in D \wedge D \cap J = \emptyset$. Applying 5.7(D) yields a g such that $\bigwedge_{j<\lambda} f^*_j < g \bmod J$, so $\bigwedge_{j<\lambda} f^*_j < g \bmod D$. On the other hand, for some $j_0 < \lambda$, $g < f^*_{j_0} \bmod J^0_{<\lambda}[a] + (a - c_\beta)$, so $g < f^*_{j_0} \bmod D$ (since $D \cap J^0_{<\lambda}[a] + (a - c_\beta) = \emptyset$) — a contradiction.]

Let $i(*) = \sup_{\beta<\alpha} i(\beta)$. Now $i(*) < \lambda$ (as $\lambda = \mathrm{cf}\,\lambda > |\alpha|$) and $c_\beta \subseteq b_{i(*)} \bmod J^0_{<\lambda}[a]$ for each $\beta < \alpha$ (because $i_1 < i_2 \Rightarrow b_{i_1} \subseteq b_{i_2} \bmod J^0_{<\lambda}[a]$) and $b_{i(*)} \in J^0_{<\lambda^+}[a]$ (by the choice of $\langle b_i : i < \lambda \rangle$ in 5.7).

(3) Let $\lambda \in \mathrm{pcf}(a)$ be minimal such that $D \cap J^0_{<\lambda^+}[a] \neq \emptyset$ and choose $b \in D \cap J^0_{<\lambda^+}[a]$. Now $(\Pi a, <_{J^0_{<\lambda}[a]+(a-b)})$ has true cofinality λ by 5.8(1). As $b \in D$, $J^0_{<\lambda}[a] \cap D = \emptyset$; we've finished the proof.

(4) Clearly $\bigcup_{\mu<\lambda} J^0_{<\lambda}[a] \subseteq J^0_{<\mu}[a]$ by 5.4(2). On the other hand, let us suppose that there is $b \in (J^0_{<\lambda}[a] - \bigcup_{\mu<\lambda} J^0_{<\lambda}[a])$. Put $J := \bigcup_{\mu<\lambda} J^0_{<\lambda}[a]$. Since $b \in J^0_{<\lambda}[a]$, for every ultrafilter D on a, if $b \in D$, then $\mathrm{tcf}(\Pi a/D) < \lambda$.

Now J is an ideal and $(\Pi a, <_J)$ is λ-directed; i.e. if $\alpha^* < \lambda$ and $\{f_\alpha : \alpha < \alpha^*\} \subset \Pi a$, then there exists $f \in \Pi a$ such that

$$(\forall \alpha < \alpha^*)(f_\alpha < f \bmod J).$$

[Why? λ is a limit, hence there is μ^* such that $\alpha^* < \mu^* < \lambda$. (W.l.o.g. $|\alpha|^+ < \mu^*$.) By 5.5, there is $f \in \Pi a$ such that $(\forall \alpha < \alpha^*)[f_\alpha < f \bmod J^0_{<\mu^+}[a])$. Since $J^0_{<\mu^*}[a] \subset J$, it is immediate that $(\forall \alpha < \alpha^*)(f_\alpha < f \bmod J)$.]

Choose an ultrafilter D on a such that $b \in D$ and $D \cap J = \emptyset$. Since $(\Pi a, <_J)$ is λ-directed and $D \cap J = \emptyset$, one has $\mathrm{tcf}(\Pi a/D) \geqq \lambda$; contradiction

(5) Easy too by 5.8(3).

5.9. Conclusion. If $|a| < \mathrm{Min}\, a$, then $\mathrm{pcf}(a)$ has a last element.

Proof. This is the minimal λ such that $a \in J^0_{<\lambda^+}[a]$. [($\lambda$ exists, since $\kappa := |\Pi a| \in \{\lambda : a \in J^0_{<\lambda^+}[a]\} \neq \emptyset$].]

5.10. Claim. Suppose $\kappa < \mathrm{Min}(a)$, for $i < \kappa$, D_i is a filter on a, E a filter on κ and $D^* = \{b \subseteq a : \{i < \kappa : b \in D_i\} \in E\}$ (a filter on a). Let $\lambda_i = \mathrm{tcf}(\Pi a, <_{D_i})$ be well defined. Let

$$\lambda^* = tcf(\Pi a, <_{D^*}), \qquad \mu = \mathrm{tcf}(\Pi \lambda_i, <_E).$$

Then $\lambda^* = \mu$ (in particular, if one is well defined, then so is the other).

Proof. Let $\langle f^i_\alpha : \alpha < \lambda_i \rangle$ be a cofinal sequence in $(\Pi a, <_{D_i})$. Define, for $g \in \Pi_{i<\kappa} \lambda_i$, $F(g) \in \Pi a$ by

$$F(g)(\theta) = \sup\{f^i_\beta(\theta) : i < \kappa, \beta = g(i)\} < \theta \qquad (\text{as } \kappa < \mathrm{Min}\, a).$$

Now for each $f \in \Pi a$, define $G(f) \in \Pi_{i<\kappa} \lambda_i$ by

$$G(f)(i) = \min\{\gamma < \lambda_i : f \leqq f^i_\gamma \bmod D_i\}$$

(it is well defined on $f \in \Pi a$ by the choice of $\langle f^i_\gamma : \gamma < \lambda_i \rangle$).

Note that for $f^1, f^2 \in \Pi a$:

$$f^1 \leqq f^2 \bmod D^* \Rightarrow B(f^1, f^2) \overset{\text{def}}{=} \{\theta \in a : f^1(\theta) \leqq f^2(\theta)\} \in D^*$$
$$\Rightarrow A(f_1, f_2) \overset{\text{def}}{=} \{i < \kappa : B(f^1, f^2) \in D_i\} \in E$$
$$\Rightarrow \bigwedge_{i \in A(f_1, f_2)} G(f^1)(i) \leqq G(f^2)(i) \quad \text{where } A(f_1, f_2) \in E$$
$$\Rightarrow G(f^1) \leqq G(f^2) \bmod E.$$

So G is a homomorphism from $(\Pi a, \leqq_{D^*})$ into $(\Pi_{i<\kappa} \lambda_i, \leqq_E)$. The range of G is a cover of $(\Pi \lambda_i, \leqq_E)$:

if $g \in \Pi_{i \leqq \kappa} \lambda_i$ then $f^i_{g(i)} \leqq F(g)$ (see definition of F) hence $g(i) \leqq [G(F(g))](i)$, hence $g \leqq G(F(g))$.

This finishes the proof.

5.11. Claim. In 5.10, if $|a|^+ < \operatorname{Min} a$, we can weaken the hypothesis $\kappa < \operatorname{Min} a$ to $\kappa < \operatorname{Min}\{\lambda_i : i < \kappa\}$.

Proof. Similar to the proof of 5.10.

We define $G : \Pi a \to \Pi_{i<\kappa} \lambda_i$ exactly as previously and also the proof of $[f^1 \leqq f^2 \bmod D^* \Rightarrow G(f^1) \leqq G(f^2) \bmod E]$ does not change.

It is enough to prove that for $g \in \Pi_{i<\kappa} \lambda_i$ for some $f \in \Pi a$, $g \leqq G(f) \bmod E$. By 5.5 $(\Pi a, <_{J^0_{\leqq\kappa}[a]})$ is κ^+-directed, hence for some $f \in \Pi a$

$(*)_1$ for $i < \kappa$, $f^i_{g(i)} < f \bmod J^0_{\leqq\kappa}[a]$.

We assume $\kappa < \lambda_i$ hence $J^0_{\leqq\kappa}[a] \subseteq J^0_{<\lambda_i}[a]$, which is disjoint from D_i (use 5.8(3)), so together with $(*)_1$

$(*)_2$ for $i < \kappa$, $f^i_{g(i)} < f \bmod D_i$.

So clearly $g < G(f)$ (more than required).

5.12. Conclusion. If $|a| < \operatorname{Min} a$, $b \subseteq \operatorname{pcf}(a)$, $|b| < \operatorname{Min} b$, then $\operatorname{pcf}(b) \subseteq \operatorname{pcf}(a)$.

§6. Normality of $\lambda \in \operatorname{pcf}(a)$ for a

6.1. Definition. (1) We say $\lambda \in \operatorname{pcf}(a)$ is normal (for a) *if*, for some $b \subseteq a$, $J^0_{<\lambda^+}[a] = J^0_{<\lambda}[a] + b$.

(2) We say $\lambda \in \operatorname{pcf}(a)$ is semi-normal (for a) if there are b_i for $i < \lambda$ such that:

(i) $i < j \Rightarrow b_i \subseteq b_j \bmod J^0_{<\lambda}[a]$

and

(ii) $J^0_{<\lambda^+}[a] = J^0_{<\lambda}[a] + \{b_i : i < \lambda\}$.

6.2. FACT. Suppose $\mathrm{Min}\, a > |a|$, $\lambda \in \mathrm{pcf}(a)$. Now:

(1) λ is semi-normal for a iff for some $F = \{f_\alpha : \alpha < \lambda\} \subset \Pi a$ for every ultrafilter D over a, F is unbounded in $(\Pi a, <_D)$ whenever $\mathrm{tcf}(\Pi a, <_D) = \lambda$.

(2) In 6.1(2) we can assume w.l.o.g. that either $b_i = b_0 \mod J^0_{<\lambda}[a]$ (so λ is normal) or $b_i \neq b_j \mod J^0_{<\lambda}[a]$ for $i < j < \lambda$.

(3) Suppose $F = \langle f_\alpha : \alpha < \lambda \rangle$ is as in (1) and is $<_{J^0_{<\lambda}[a]}$-increasing. Then λ is normal *iff* F has a $<_{J^0_{<\lambda}[a]}$-least upper bound $g \in \Pi_{\theta \in a}(\theta + 1)$ and then $\{\theta \in a : g(\theta) = \theta\}$ generates $J^0_{<\lambda^+}[a]$.

PROOF. Left to the reader. Use 5.7, 5.8(3) for (1), (2).

We shall give some sufficient conditions for this normality.

6.3. DEFINITION. For given regular λ, $\theta < \mu < \lambda$, $S \subseteq \lambda$, $\sup S = \lambda$.

(1) We call $\bar{A} = \langle A_\alpha : \alpha < \lambda \rangle$ a continuity condition for (S, μ, θ) if: $A_\alpha \subseteq \alpha$, $|A_\alpha| < \mu$ for $\alpha \in S$, $[\delta \in S \Rightarrow \mu > \mathrm{cf}\,\delta \geqq \theta]$ and $[\beta \in A_\alpha \Rightarrow A_\beta = A_\alpha \cap \beta]$, $[\delta \in S \Rightarrow \delta = \sup A_\delta]$.

(2) We say $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ obeys $\bar{A} = \langle A_\alpha : \alpha < \lambda \rangle$ if:

(a) for $\beta \in A_\alpha$, $\bigwedge_{\theta \in a} f_\beta(\theta) < f_\alpha(\theta)$,

(b) if $\alpha \in S$ then $f_\alpha(\theta) = \sup_{\beta \in A_\alpha} f_\beta(\theta)$ for every $\theta \in a$.

(3) If $\theta = \aleph_0$ we omit it; (S, a) stands for $(S, \mathrm{Min}\, a, |a|^+)$, (λ, μ, θ) stands for "(S, μ, θ) for some stationary $S \subseteq \lambda$"; similarly (λ, a).

(4) We add the adjective "weak" if "$\beta \in A_\alpha \Rightarrow A_\beta = A_\alpha \cap \beta$" is replaced by "$\alpha \in S \;\&\; \beta \in A_\alpha \Rightarrow (\exists \gamma < \alpha)[A_\alpha \cap \beta \subseteq A_\gamma]$".

(5) $I^g[\lambda] \overset{\mathrm{def}}{=} \{S \subseteq \lambda$: there is a sequence $\langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$ such that $\mathcal{P}_\alpha$ is a family of $< \lambda$ subsets of λ, and for every $\delta \in S$ for some unbounded $A \subseteq \delta$, $\mathrm{otp}\, A < \delta$ and $[\alpha \in A \Rightarrow A \cap \alpha \in \bigcup_{\beta < \delta} \mathcal{P}_\beta]\}$.

(6) $I^{wg}_{\mu,\theta}[\lambda] = \{S \subseteq \lambda$: there is a sequence $\langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$ such that $\mathcal{P}_\alpha$ is a family of $< \lambda$ subsets of λ each of power $< \mu$ and for every $\delta \in S$ for some unbounded $A \subseteq \delta$, $(\forall \alpha \in A)(\exists x \in \bigcup_{\beta < \delta} \mathcal{P}_\beta)[A \cap \alpha \subseteq x]\}$.

(7) Stationary members of $I^g[\lambda]$ are called good stationary sets; similarly, stationary members of $I^{wg}_{\mu,\theta}[\lambda]$ are called weakly good stationary sets. Again $I^{wg}_\mu[\lambda]$ is $I^{wg}_{\mu,\aleph_0}[\lambda]$.

For definitions and proofs see [Sh 88], AP Lemma 2, [Sh 300a], Ch. III, §6, [Sh 351] 4.1.

6.4. FACT. (1) There is a [weak] continuity condition $\bar{A}$ for (λ, a) *iff* there is stationary S such that $S \subseteq \{\delta < \lambda : |a| < \operatorname{cf} \delta < \operatorname{Min} a\}$ is in $I^g[\lambda]$ [in $I^{\mathrm{wg}}_{\operatorname{Min} a}[\lambda]$].

(2) If $\lambda = \mu^+$, $\operatorname{cf} \mu = \mu > \aleph_0$, then $\{\delta < \lambda : \operatorname{cf}(\delta) < \mu\}$ is in $I^g[\lambda]$.

(3) If $\lambda = \mu^+$, $\theta < \operatorname{cf} \mu$, then $\{\delta < \lambda : \operatorname{cf} \delta = \theta\}$ contains a stationary set from $I^{\mathrm{wg}}_{\kappa,\theta}[\lambda]$ for some $\kappa < \mu$.

(4) If $\lambda = \mu^+$, $\mu \rightarrow (\theta)^2_{\operatorname{cf} \mu}$, then there are $\kappa < \mu$ and a stationary $S \subseteq \{\delta < \lambda : \operatorname{cf} \delta = \theta\}$ which is in $I^{\mathrm{wg}}_{\kappa,\theta}[\lambda]$.

6.5. FACT. Suppose $\bar{A}$ is a weak continuity condition for (S, a), $f_\alpha \in \Pi a$ for $\alpha < \lambda$, $\operatorname{Min} a > |a|^+$, $\lambda = \operatorname{cf} \lambda > |a|$. Then:

(1) We can find $\langle f'_\alpha : \alpha < \lambda \rangle$ obeying $\bar{A}$, $f'_\alpha \in \Pi a$, such that

(i) for $\alpha \in \lambda - S$, $f_\alpha \leqq f'_\alpha$,

(ii) for every α, $f_\alpha \leqq f'_{\alpha+1}$.

(2) Suppose $\langle f'_\alpha : \alpha < \lambda \rangle$ obeys $\bar{A}$ and satisfies (i). If $g_\alpha \in \Pi a$, $\langle g_\alpha : \alpha < \lambda \rangle$ obeys $\bar{A}$ and $\bigwedge_\alpha g_\alpha \leqq f_\alpha$, then $\bigwedge_\alpha g_\alpha \leqq f'_\alpha$.

(3) We can add in (1)

(iii) if $\langle f''_\alpha : \alpha < \lambda \rangle$ obeys $\bar{A}$, $f''_\alpha \in \Pi a$, and it satisfies (i), *then* for every α, $f'_\alpha \leqq f''_\alpha$.

PROOF. Easy.

6.6. LEMMA. *Suppose $f_\alpha \in \Pi a$ for $\alpha < \lambda$, λ regular, $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ obeys some $\bar{A} = \langle A_\alpha : \alpha < \lambda \rangle$ which is a weak continuity condition for (λ, a), and $\bar{f}$ is $J^0_{<\lambda}[a]$-increasing (so $\lambda \geqq \operatorname{Min}(a)$).*

(a) *$\langle f_\alpha : \alpha < \lambda \rangle$ has a $<_{J^0_{<\lambda}[a]}$-least upper bound $g \in \Pi_{\theta \in a}(\theta + 1)$.*

(b) *$b_g \in J^0_{<\lambda^+}[a] - J^0_{<\lambda}[a]$ where $b_g \overset{\mathrm{def}}{=} \{\theta \in a : g(\theta) = \theta\}$.*

(c) *Letting $\mu_\theta = \operatorname{cf}(g(\theta))$, we have that $(\Pi \mu_\theta, <_{J^0_{<\lambda}[a]})$ has true cofinality λ and $\mu_\theta \leqq \theta$.*

PROOF. See [Sh 282], Lemma 14 for (a).

6.7. CLAIM. Suppose:

(a) $f_\alpha \in \Pi a$ for $\alpha < \lambda$, $\lambda \in \operatorname{pcf}(a)$ and $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ is $<_{J^0_{<\lambda}[a]}$-increasing.

(b) $\bar{f}$ satisfies $\bar{A}$, a weak continuity condition for (S, a), $\lambda = \sup S$ (hence $\lambda \geqq \operatorname{Min}(a) > |a|^+$).

(c) J is an ideal of $\mathscr{P}(a)$ extending $J^0_{<\lambda}[a]$, and $\langle f_\alpha / J : \alpha < \lambda \rangle$ is cofinal in $(\Pi a, <_J)$ (e.g. $J = J^0_{<\lambda}[a] + (a - b)$, $b \in J^0_{<\lambda^+}[a] - J^0_{<\lambda}[a]$).

(d) $\langle f'_\alpha : \alpha < \lambda \rangle$ satisfies (a), (b) above.

(e) $f_\alpha \leqq f'_\alpha$ for $\alpha < \lambda$, alternatively: $\langle f'_\alpha : \alpha < \lambda \rangle$ satisfies (c).

Then $\{\delta < \lambda : \text{if } \delta \in S \text{ then } f'_\delta = f_\delta \bmod J\}$ contains a club of λ.

PROOF. Not hard.

6.8. LEMMA. *Suppose* Min $a > |a|^+$, $\lambda = \mathrm{cf}\,\lambda \in \mathrm{pcf}(a)$ *and there is a* good stationary set $\subseteq \{\delta < \lambda : |a| < \mathrm{cf}\,\delta < \mathrm{Min}\, a\}$ *or at least a* weakly good stationary set $\subseteq \{\delta < \lambda : |a| < \mathrm{cf}\,\delta < \mathrm{Min}\, a\}$. Then λ *is normal for* a.

PROOF. Let $\bar{A}$ be a weak continuity condition for (S, a) for some S, where S is a stationary subset of $\{\delta < \lambda : |a| < \mathrm{cf}\,\delta < \mathrm{Min}\, a\}$. We assume λ is not normal for a and eventually get a contradiction. By 6.2, 6.6 λ is not semi-normal for a. Let us define by induction on $\zeta \leqq |a|^+$, $\bar{f}^\zeta = \langle f^\zeta_\alpha : \alpha < \lambda \rangle$ and D_ζ, such that:

(I) (i) $f^\zeta_\alpha \in \Pi a$,

(ii) $\alpha < \beta \Rightarrow f^\zeta_\alpha < f^\zeta_\beta \bmod J^0_{<\lambda}[a]$,

(iii) $\bar{f}^\zeta$ obeys $\bar{A}$,

(iv) for $\xi < \zeta \leqq |a|^+$ and $\alpha < \lambda : f^\xi_\alpha \leqq f^\zeta_\alpha$;

(II) (i) D_ζ is an ultrafilter on a such that $\mathrm{cf}(\Pi a / D_\zeta) = \lambda$,

(ii) $\langle f^\zeta_\alpha / D_\zeta : \alpha < \lambda \rangle$ is not cofinal in $\Pi a / D_\zeta$,

(iii) $\langle f^{\zeta+1}_\alpha / D_\zeta : \alpha < \lambda \rangle$ is cofinal in $\Pi a / D_\zeta$,

(iv) $f^{\zeta+1}_0 / D_\zeta$ is above $\{ f^\zeta_\alpha / D_\zeta : \alpha < \lambda \}$.

For $\zeta = 0$: No problem. [Use 6.5 and 6.2.]

For ζ *limit*: Let $g^\zeta_\alpha \in \Pi a$ be defined by $g^\zeta_\alpha(\theta) = \sup_{\xi < \zeta} f^\xi_\alpha(\theta)$, which belongs to Πa as $|a|^+ < \mathrm{Min}(a)$. Now use 6.5 and get $\langle f^\zeta_\alpha : \alpha < \lambda \rangle$ obeying $\bar{A}$, $[\zeta \in \lambda - S \Rightarrow g^\zeta_\alpha \leqq f^\zeta_\alpha]$, $[g^\zeta_\alpha \leqq f^\zeta_{\alpha+1}]$. Use 6.5 to find an appropriate D_ζ. Now $\langle f^\zeta_\alpha : \alpha < \lambda \rangle$ and D_ζ are as required.

For $\zeta = \xi + 1$: By 6.2(1) there is an ultrafiler D_ξ on a such that $\mathrm{tcf}(\Pi a, <_{D_\xi}) = \lambda$ and $\{ f^\xi_\alpha : \alpha < \lambda \}$ is bounded in $(\Pi a, <_{D_\xi})$. Let $\langle h^\xi_\alpha : \alpha < \lambda \rangle$ be cofinal in $(\Pi a, <_{D_\xi})$ and w.l.o.g. $f^\xi_\alpha \leqq h^\xi_0 \bmod D_\zeta$. We get D_ζ and $\langle f^\zeta_\alpha : \alpha < \lambda \rangle$ by 6.2 and 6.5 for $\langle h^\xi_\alpha : \alpha < \lambda \rangle$.

Now for each $\zeta < |a|^+$ we apply 6.7 for $\langle f^{\zeta+1}_\alpha : a < \lambda \rangle$, $\langle f^{|a|^+}_\alpha : \alpha < \lambda \rangle$, $J = P(a) \setminus D_\zeta$. We get a club C_ζ of λ such that:

(*) $\alpha \in S \cap C_\zeta \Rightarrow f^{\zeta+1}_\alpha = f^{|a|^+}_\alpha \bmod D_\zeta$.

So $\bigcap_{\zeta < |a|^+} C_\zeta$ is a club of λ since $|a|^+ < \lambda$, so we can choose $\alpha \in S \cap \bigcap_{\zeta < |a|^+} C_\zeta$. Let $c_\zeta = \{\theta \in a : f^\zeta_\alpha(\theta) = f^{|a|^+}_\alpha(\theta)\}$. By (*), $c_{\zeta+1} \in D_\zeta$; by (II)(ii), (iv) $c_\zeta \notin D_\zeta$, hence $c_\zeta \neq c_{\zeta+1}$. On the other hand, by (I) (iv), $\langle c_\zeta : \zeta < |a|^+ \rangle$ is $\subseteq$-increasing and by the previous sentence it is strictly $\subseteq$ -increasing; contradition.

6.9. CLAIM. Suppose $\mathrm{Min}(a) > |a|^+$, $\mu = \mathrm{cf}\,\mu < \lambda \in \mathrm{pcf}(a)$. Then for some $\kappa_\theta = \mathrm{cf}\,\kappa_\theta < \theta$ (for $\theta \in a$) we have $(\Pi_{\theta\in a}\kappa_\theta, <_{J^0_{<\mu}[a]})$ has true cofinality μ, provided that

(*) μ has a weakly good stationary set $S \subseteq \{\delta < \mu : |a| < \mathrm{cf}\,\delta < \mathrm{Min}\,a]$.

PROOF. Easy, by 6.6, 6.5.

6.10. CLAIM. Suppose the assumptions (a), (c), (d), (e) of 6.7 hold and

(b)′ $\bar{f}$ obeys $\bar{A}$, $\bar{A}$ a continuity condition for $(S, \kappa, \aleph_0)(\lambda = \sup S)$.

(f) J is κ-complete, $\kappa = \mathrm{cf}\,\kappa > \mathrm{cf}(\delta)$ for every $\delta \in S$.

Then for some club C of λ

$$\delta \in S \cap C \Rightarrow f'_\alpha = f_\alpha \bmod J.$$

PROOF. Not hard. (See 6.7.)

6.11. LEMMA. *Suppose* $\mathrm{Min}(a) > |a|^+$, $\lambda \in \mathrm{pcf}(a)$. *Then there is* $b \subseteq a$ *such that* $b \in J^0_{<\lambda^+}[a]$ *and*

(*) *for every* $c \in J^0_{<\lambda^+}[a]$ *there are* $b_n \in J^0_{<\lambda}[a]$ *for* $n < \omega$ *such that* $c \subseteq b \cup \bigcup_{n<\omega} b_n$.

PROOF. Let $S = \{\delta < \lambda : \mathrm{cf}\,\delta = \aleph_0$ or δ is a successor ordinal$\}$. We can *easily* find a continuity condition $\bar{A} = \langle A_\alpha : \alpha < \lambda\rangle$, for $(S, \aleph_1, \aleph_0)$ such that, for limit $\delta \in S$, A_δ is an unbounded subset of δ of order type ω, and for non-limit $\alpha \in S$, A_α is finite. Here is how one finds the continuity condition.

We prove by induction on $\alpha \leqq \lambda$ the existence of a continuity condition $\bar{A}^\alpha = \langle A^\alpha_i : i \in \alpha \cap S\rangle$:

(1) $\alpha \leqq \omega + 1$: let $A_i = i$ for $i < \alpha$.

(2) Not (1) and $\alpha = \beta + \gamma$ where $\beta < \alpha$, $\gamma < \alpha$, $\mathrm{cf}(\beta) \neq \aleph_0$.

Let

$$A^\alpha_i = \begin{cases} A^\beta_i, & i \in \beta \cap S \\ \beta + A^\gamma_j, & i \in \alpha \cap S \setminus \beta, \quad i - \beta = j \end{cases}$$

where $\beta + A = \{\beta + \zeta : \zeta \in A\}$.

(3) Not (1), (2) and $\alpha = \beta$, $\mathrm{cf}(\beta) = \aleph_0$ or $\alpha = \beta + 1$, $\mathrm{cf}\,\beta = \aleph_0$.

Let $\beta = \bigcup_{n<\omega}\alpha_n$, where $0 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots$, $\mathrm{cf}(\alpha_{n+1}) \neq \aleph_0$ (e.g. α_{n+1} successor),

$$A^\alpha_\beta = \{\alpha_n : n < \omega\}\ [\mathrm{cf}(\beta) < \alpha],$$

$$A^\alpha_{\alpha_n} = \{\alpha_m : m < n\},$$

if $\alpha_n < \gamma < \alpha_{n+1}$ let $A^\alpha_\gamma := \alpha_{n+1} + A^{\alpha_{n+1}-(\alpha_n+1)}_{\gamma-(\alpha_n+1)}$.

(4) Not (1), (2), (3), $\alpha > \mathrm{cf}(\alpha) > \aleph_0$.

Let $\kappa = \mathrm{cf}(\alpha)$. Let $\langle \alpha_i : i < \kappa \rangle$ be inceasing continuous, $\bigcup_{i<\kappa} \alpha_i =: \alpha$, $\alpha_0 = 0$, $\mathrm{cf}(\alpha_{i+1}) \neq \aleph_0$.

We define for each $\langle A^\alpha_\gamma : \alpha_i < \gamma < \alpha_{i+1} \rangle$ by the induction hypothesis

$$A^\alpha_\gamma = (\alpha_i + 1) + A^{\alpha_{i+1} - (\alpha_i + 1)}_{\gamma - (\alpha_i + 1)} \qquad \text{for } \alpha_i < \gamma < \alpha_{i+1},$$

$$A^\alpha_{\alpha_i} = \{\alpha_j : j \in A^\kappa_i\}.$$

(5) $\alpha = \mathrm{cf}\,\alpha > \aleph_0$.

Call $\alpha = \kappa$. Choose $\langle \alpha_i : i < \kappa \rangle$ increasing continuous, $\bigcup_{i<\kappa} \alpha_i = \alpha$, $\alpha_0 = 0$, $\mathrm{cf}(\alpha_{i+1}) > \aleph_0$† and $\alpha_{i+1} > (\omega + \omega) + (\alpha_i + \alpha_i) + \omega$. So $E_i = \{\delta + 1 : \delta \text{ limit}, \alpha_i < \delta + 1 < \alpha_{i+1}\}$ has power $\geqq |\alpha_i|$.

Let g_i be a function from E_i onto $\bigcup_{j<i} E_j$.

We define $h : \kappa \to \kappa$,

$$h(\alpha) = \begin{cases} \alpha + 1, & \alpha \text{ successor}, \\ \alpha, & \text{otherwise}. \end{cases}$$

Choose A^α as follows: for $\alpha_i < \gamma < \alpha_{i+1}$, let $B^\alpha_\gamma = (\alpha_i + 1) + A^{\alpha_{i+1} - (\alpha_i + 1)}_{\gamma - (\alpha_i + 1)}$, $A^\alpha_{h(\gamma)} = h(B^\alpha_\gamma)$. So we have defined A^α_β for $\beta \in \bigcup_i ((\alpha_i, \alpha_{i+1}) \setminus E_i)$.

For $\gamma \in E_i$ we define A^α_β by induction on γ:

$$i = 0, \qquad A^\alpha_\gamma = 0;$$

$$i > 0, \qquad A^\alpha_\gamma = \{h_i(\gamma)\} \cup A^\alpha_{h_i(\gamma)}.$$

Lastly for $\gamma \in \{\alpha_i : i < \kappa\}$, if $\mathrm{cf}(\alpha_i) = \aleph_0$, then $\mathrm{cf}(i) = \aleph_0$. So there are $\langle j_n : n < \omega \rangle$:

$$0 = j_0 < j_1 < \cdots$$

and

$$\bigcup j_n = i.$$

Choose inductively $\gamma^i_n \in E_{j_n}$, $h(\gamma^i_{n+1}) = \gamma^i_n$. So

$$A^\alpha_{\gamma^i_n} = \{\gamma^i_0, \ldots, \gamma^i_{n-1}\} \quad \text{and} \quad A^\alpha_{\alpha_i} \stackrel{\text{def}}{=} \{\gamma^i_n : n < \omega\}.$$

Now after this digression, we return to the proof of 6.11. The proof is the same as that of 6.8, using 6.10 instead of 6.7, applied to $J \stackrel{\text{def}}{=} J^1_{<\lambda}[a] =$

† We assume $\kappa > \aleph_1$; if $\kappa = \aleph_1$, the changes are small.

$\{\bigcup_n b_n : b_n \in J^0_{<\lambda}[a]$ for $n < \omega\}$ — which is an $\aleph_1$-complete ideal (we use J instead of $J^0_{<\lambda}[a]$).

6.12. CONCLUSION. Suppose $\mathrm{Min}\, a > |a|^+$.

(1) We can find $\langle b_\lambda : \lambda \in \mathrm{pcf}(a)\rangle$ such that:

(i) $b_\lambda \in J^0_{<\lambda^+}[a] - J^0_{<\lambda}[a]$,

(ii) every member of $J^0_{<\lambda}[a]$ is included in some $\bigcup_{n<\omega} b_{\lambda_n}$, for some $\lambda_n < \lambda$.

(2) If every $\lambda \in \mathrm{pcf}(a)$ is normal for a, then we can replace (ii) above by

(ii)′ $J^0_{<\lambda}[a]$ is a generated by $\{b_\mu : \mu \in \lambda \cap \mathrm{pcf}(a)\}$.

6.13. FACT. (1) Suppose $|\mathrm{pcf}(a)|^{\aleph_0} < \mathrm{Min}\, a$ (or $(*)_2$ of 9.1). If $\lambda \in \mathrm{pcf}(a)$, and

$(*)_\kappa$ [if $\mu_i \in \mathrm{pcf}(a) \cap \lambda$ for $i < \alpha < \kappa$ then $\Pi_{i<\alpha} \mu_i < \lambda$],

then $J^0_{<\lambda}[a]$ is a κ-complete ideal.

(2) If in (1) $\kappa \geqq \aleph_1$, then λ is normal for a.

6.13A. REMARK. To prove 6.13, we rely here on a later Theorem (9.1), so till 9.1 we cannot use 6.13.

PROOF. (1) Suppose $J^0_{<\lambda}[a]$ is not κ-complete, then there are $\alpha < \kappa$ and $b_i \in J^0_{<\lambda}[a]$ for $i < \alpha$ and $\bigcup_{i<\alpha} b_i \notin J^0_{<\lambda}[a]$. W.l.o.g. α is minimal, hence $\alpha = \mathrm{cf}(\alpha)$ and w.l.o.g. $[i < j < \alpha \Rightarrow b_i \subseteq b_j]$. By 9.1(1) for some $c \subseteq \bigcup_{i<\alpha} \mathrm{pcf}(b_i)$, $|c| \leqq |\alpha|$ and $\lambda \in \mathrm{pcf}(c)$. Now $b_i \in J^0_{<\lambda}[a]$ hence $\max \mathrm{pcf}(b_i) < \lambda$, hence c is a set of $< \kappa$ regular cardinals, each $< \lambda$ and from $\bigcup_{i<\alpha} \mathrm{pcf}(b_i) \subseteq \mathrm{pcf}(a)$. By $(*)_\kappa$ we get a contradiction.

(2) By 6.11 and the first part.

6.14. LEMMA. *Suppose* $|\mathrm{pcf}(a)|^{\aleph_0} < \mathrm{Min}\, a$. *Then every* $\lambda \in \mathrm{pcf}(a)$ *is normal for* a.

PROOF. W.l.o.g. $a = \mathrm{pcf}(a)$. [Just prove that if $a \subseteq b$, $|b| < \min(b)$ and λ is normal for b, then λ is normal for a.]

We prove by induction on λ, and for a fixed λ by induction on θ, that

$(*)$ if $|\mathrm{pcf}(a)|^{\aleph_0} < \mathrm{Min}\, a$, $\lambda \in \mathrm{pcf}(a)$, $\theta = \sup\{\mu^+ : \mu \in \mathrm{pcf}(a), \mu < \lambda\}$, *then* λ is normal for a.

Case I: $\theta = \mu^+$.

Necessarily $\mu \in \mathrm{pcf}(a)$. By the induction hypothesis for some $b_\mu \subseteq a$, $J^0_{<\mu^+}[a] = J^0_{<\mu}[a] + b_\mu$.

Now $\lambda \notin \mathrm{pcf}(b_\mu)$ so $\lambda \in \mathrm{pcf}(a - b_\mu)$, and by the choice of b_μ and 5.8(3), $\mu \notin \mathrm{pcf}(a - b_\mu)$, so $\theta^* \stackrel{\mathrm{def}}{=} \sup(\lambda \cap pcf(a - b_\mu)) \leqq \mu$. So we can apply the induction hypothesis on λ, θ^*, $a - b_\mu$ and get that λ is normal for $a - b_\mu$. As $\lambda \notin \mathrm{pcf}(b_\mu)$, by 5.3(2), λ is normal for a as required.

Case II: θ is a limit cardinal.

Remember $a = \mathrm{pcf}(a)$.

Let $c = \theta \cap \mathrm{pcf}(a)$, $J_c^{\mathrm{bd}} = \{c' \subseteq c; c' \text{ is bounded in } c\}$. Now if D is an ultrafilter on c disjoint from J_c^{bd}, then $\mathrm{tcf}(\Pi c, <_D)$ is necessarily $\geqq \theta$ (by 5.3(4)), but it belongs to $\mathrm{pcf}(c)$ which, by 5.11, is a subset of $\mathrm{pcf}(a)$, hence by assumption it is $\geqq \lambda$. We conclude $D \cap J^0_{<\lambda}[a] = \emptyset$. As this holds for every such D we know $J^0_{<\lambda}[a] \restriction c \subseteq J_c^{\mathrm{bd}}$, so easily $J^0_{<\lambda}[a] \subseteq J_c^{\mathrm{bd}}$.

Case IIa: $\mathrm{cf}(\theta) > \aleph_0$.

J_c^{bd} is $\aleph_1$-complete, so by the argument of 6.11 there is $b^* \subseteq c$ such that:

(i) $b^* \in J^0_{<\lambda^+}[a]$,

(ii) $(\forall b' \in J^0_{<\lambda^+}[a])(b' - b \in J_c^{\mathrm{bd}})$.

We claim

(∗) for some $\sigma \in c$, $\lambda \notin \mathrm{pcf}(c - \sigma - b^*)$.

[If not, for every $\sigma \in c$ there is $b_\sigma \in J^0_{<\lambda^+}[a] - J^0_{<\lambda}[a]$, $b_\sigma \subseteq c$, $b_\sigma \cap b^* = \emptyset$ and $\mathrm{Min}\, b_\sigma \geqq \sigma$. By 5.8(2) there is $b' \subseteq c$, $b' \in J^0_{<\lambda^+}[a]$ such that $\sigma \in c \Rightarrow b_\sigma \subseteq b' \bmod J^0_{<\lambda}[a]$. As $b_\sigma \subseteq c - b^*$, $\mathrm{Min}\, b_\sigma \geqq \sigma$ we have $b' - b^* \subseteq c$ unbounded in c, and contradicting (ii) above.]

Now λ is normal for b^* (as $b^* \in J^0_{<\lambda^+}[a]$). Also $\lambda \notin \mathrm{pcf}(c - \sigma - b^*)$ (by (∗)) hence λ is normal for $c - \sigma - b^*$; moreover, by the induction hypothesis applied to $\lambda, c \cap \sigma$ λ is normal for $c \cap \sigma$. Together (see 5.8(3)) λ is normal for c. Also, as $\mathrm{Min}(a - \lambda) = \lambda$, λ is normal for $a - \lambda$ so it is normal for a.

Case IIb: $\mathrm{cf}\,\theta = \aleph_0$.

Using $|\mathrm{pcf}\, a|^{\aleph_0} = |a|^{\aleph_0} < \mathrm{Min}\, a < \lambda$. Apply 5.8(2) to $\{b \subseteq c : |b| = \aleph_0, b \in J^0_{<\lambda^+}[a] - J^0_{<\lambda}[a]\}$ and proceed as in Case IIa.

§7. Getting better representations: generating sequences and cofinality systems

We can replace systematically normal by semi-normal and b_λ by $\langle b_i^\lambda : i < \lambda \rangle$ as in Definition 6.1, by avoiding it to ease the reading.

7.1. Definition. (1) We say $\langle b_\lambda ; \lambda \in c \rangle$ is a generating sequence for a if:

(i) $b_\lambda \subseteq a$, $c \subseteq \mathrm{pcf}\, a$,

(ii) $J^0_{<\lambda^+}[a] = (J^0_{<\lambda}[a]) + b_\lambda$.

(2) Let $J^{1,\kappa}_{<\lambda}[a]$ be the κ-complete ideal on $\mathscr{P}(a)$ generated by $J^0_{<\lambda}[a]$.

(3) Let $\mathrm{pcf}^{1,\kappa}(a) = \{\lambda \in \mathrm{pcf}(a) : J^{1,\kappa}_{<\lambda}[a] \neq J^{1,\kappa}_{<\lambda^+}[a]\}$ (See 7.1(6).)

(4) We say $\langle b^a_\lambda : \lambda \in c \rangle$ is a weak generating sequence for a if

(i) $b^a_\lambda \subseteq a$, $b^a_\lambda \notin J^0_{<\lambda}[a]$, $b^a_\lambda \in J^0_{<\lambda^+}[a]$,

(ii) $c \subseteq \mathrm{pcf}(a)$.

(5) We say $\langle b^a_\lambda : \lambda \in c \rangle$ is a κ-almost generating sequence for a if (i), (ii) of (4) hold and

(iii) $J^{1,\kappa}_{<\lambda^+}[a] = (J^{1,\kappa}_{<\lambda}[a]) + b^a_\lambda$.

(6) In (2), (3), (5) if $\kappa = \aleph_1$, we omit it.

(7) We call $\bar{b} = \langle b_\lambda : \lambda \in c \rangle$ smooth if $\theta \in b_\lambda \Rightarrow b_\theta \subseteq b_\lambda$.

7.2. Fact. Let $|a|^+ < \mathrm{Min}\, a$.

(1) $\lambda \in \mathrm{pcf}^1(a)$ iff for some $\aleph_1$-complete ideal J on a, $\lambda = \mathrm{tcf}(\Pi a, <_J)$.

(2) There is an almost generating sequence $\langle b_\lambda : \lambda \in \mathrm{pcf}^1(a) \rangle$ for a.

(3) There is a generating sequence $\langle b_\lambda : \lambda \in \mathrm{pcf}(a) \rangle$ for a if at least one of the following holds:

(i) $2^{|a|} < \mathrm{Min}\, a$,

(ii) $|\mathrm{pcf}(a)|^{\aleph_0} < \mathrm{Min}\, a$,

(iii) every $\lambda \in \mathrm{pcf}(a)$ has a (λ, a)-weakly good stationary set (see Definition 6.3)

(4) An $\aleph_0$-almost generating sequence is a generating sequence.

(5) Suppose $\bar{b} = \langle b_\lambda : \lambda \in \mathrm{pcf}(a) \rangle$ is a generating sequence, and $b \subseteq a$, $b = \mathrm{pcf}(b)$, *then* for some finite $d \subseteq b$, $b \subseteq \bigcup_{\theta \in d} b_\theta$.

Proof. (1) If $\lambda \in \mathrm{pcf}^1(a)$, i.e. $\lambda \in \mathrm{pcf}^{1,\aleph_1}(a)$ (see 7.1(6)), this means $J^1_{<\lambda}[a] \neq J^1_{<\lambda^+}[a]$, i.e. $J^{1,\aleph_1}_{<\lambda}[a] \neq J^{1,\aleph_1}_{<\lambda^+}[a]$. So choose $b \in J^1_{<\lambda^+}[a]$, $b \notin J^1_{<\lambda}[a]$, and let $J = J^1_{<\lambda}[a] + (a - b)$.

The other direction is trivial too. (Use 5.8(3) and note that $J^1_{<\lambda}[a] \neq J^1_{<\lambda^+}[a]$ iff $J^1_{<\lambda}[a] \not\supseteq J^0_{<\lambda^+}[a]$.)

(2) By 6.11.

(3) We can assume a is infinite.

If (i), then as $|\mathrm{pcf}(a)| \leqq 2^{|a|}$ (by 5.8(5)) then $|\mathrm{pcf}(a)|^{\aleph_0} \leqq (2^{|a|})^{\aleph_0} = 2^{|a|} < \mathrm{Min}\, a$, so (ii) holds.

If (ii) holds, use 6.14.

If (iii) holds, use 6.8.

(4) Check.

(5) If not, then $I = \{b \cap \bigcup_{\theta \in d} b_\theta : d \subseteq b, d \text{ finite}\}$ is a family of subsets of b,

closed under union, $b \notin I$, hence there is an ultrafilter D on b disjoint from I. Let $\theta \stackrel{\text{def}}{=} \text{cf}(\Pi b/D)$; as $b = \text{pcf}(b)$ necessarily $\theta \in b$. Let D' be the ultrafilter on a which D generates, clearly $\theta = \text{cf}(\Pi a/D')$; by 5.8(3), $b_\theta \in D'$, hence $b \cap b_\theta \in D$, contradicting the choice of D.

7.3. Definition. (1) For a weak generating sequence $\bar{b} = \langle b_\lambda : \lambda \in c \rangle$ for a we say $\bar{f} = \langle \langle f_{\lambda,\alpha} : \alpha < \lambda \rangle : \lambda \in c \rangle$ is a cofinal sequence for $(a, \bar{b})$ if

(i) $\langle f_{\lambda,\alpha} : \alpha < \lambda \rangle$ is strictly increasing and cofinal in $(\Pi(a \cap \lambda^+), <_{J^0_{<\lambda}[a] + (a - b_\lambda)})$.

(2) $\bar{f}$ is continuous if [* continuous]

(ii) if $\delta < \lambda$, $|a| < \text{cf}\,\delta < \text{Min}\, a$ then

$$f_{\lambda,\delta} = f^0_{\lambda,\delta} \quad \left[f_{\lambda,\delta}(\theta) = \bigcup_n f^n_{\lambda,\delta}(\theta) \right]$$

where $f^n_{\lambda,\delta}(\theta)$ is defined by induction on $n < \omega$,

$$f^0_{\lambda,\delta}(\theta) = \text{Min}\left\{ \bigcup_{\alpha \in C} f_{\lambda,\alpha}(\theta) : C \subseteq \delta \text{ is a club} \right\},$$

$$\rho^{n+1}_{\lambda,\delta}(\theta) = \sup\{ f_{\mu,\alpha}(\theta) : \theta \leqq \mu < \lambda, \mu \in a, \alpha = f^n_{\lambda,\delta}(\mu)\} \cup \{ f^n_{\lambda,\delta}(\theta)\}.$$

(3) $\bar{f}$ is nice if it is * continuous and in addition:

(iii) if $\delta < \lambda$, then

$$\theta \in a \;\&\; \sigma \in a \cap \theta^+ \Rightarrow f_{\theta, f_{\lambda,\delta}(\theta)}(\sigma) \leqq f_{\lambda,\delta}(\sigma),$$

except possibly when $|a| < \text{cf}\,\delta < \text{Min}\, a$, $\text{cf}[f_{\lambda,\delta}(\sigma)] \neq \text{cf}\,\alpha$.

7.4. Fact. Assume $|a| < \text{Min}\, a$.

(1) For every weak generating sequence $\bar{b}$ for a, some $\bar{f}$ is a * continuous cofinal sequence for $(a, \bar{b})$.

(2) If $\langle \langle f_{\lambda,\alpha} : \alpha < \lambda \rangle : \lambda \in \text{pcf}(a) \rangle$ is a cofinal sequence for $(a, \bar{b})$, $\bar{b}$ is a generating sequence for a with domain $\text{pcf}(a)$, *then*

$(*)_2$ for every $g \in \Pi a$ there are $n < \omega$, $\lambda_0 > \lambda_1 > \cdots > \lambda_n$ from $\text{pcf}(a)$ and $\alpha_l < \lambda_l$ for $l \leqq n$ such that

$$g \leqq \text{Max}\{ f_{\lambda_l,\alpha_l} : l \leqq n \}.$$

(3) In (1), if $\bar{b}$ is only a κ-almost generating sequence for a (so its domain $\supseteq \text{pcf}^{1,\kappa}(a)$), *then*

$(*)_3$ for every $g \in \Pi a$ there is a set $b \subseteq \operatorname{pcf} a$ of power $< \kappa$ and $\langle \alpha_\theta : \theta \in b \rangle$ such that $\alpha_\theta < \theta$ and

$$g < \sup\{f_{\lambda,\alpha_\lambda} : \lambda \in b\};$$

in fact $\forall \theta \in a \bigvee_{\lambda \in b} g(\theta) < f_{\lambda,\alpha_\lambda}(\theta)$.

Proof. (1) We define $\langle f_{\lambda,\alpha} : a < \lambda \rangle$ for each $\lambda \in c$. By 5.5 there is $\langle f^*_{\lambda,\alpha} : \alpha < \lambda \rangle$, $<_J$-increasing, where $J = (J^0_{<\lambda}[a] + (a - b_\lambda)) \restriction (a \cap \lambda^+)$ and cofinal in $(\Pi(a \cap \lambda^+), <_J)$. We now choose $f_{\lambda,\alpha}$ by induction on α such that:

(a) for α nonlimit, $f^*_{\lambda,\alpha} \leqq f_{\lambda,\alpha} \in \Pi(a \cap \lambda^+)$,

(b) for $\beta < \alpha$, $f_{\lambda,\beta} <_J f_{\lambda,\alpha}$,

(c) if α is limit, $|a| < \operatorname{cf} \alpha < \operatorname{Min} a$, then (ii) of 7.3(2) holds.

The only problematic point is, why, if $\alpha = \delta$, $|a| < \operatorname{cf} \delta < \operatorname{Min} a$, if we define $f_{\lambda,\delta}$ as required in (c), then it satisfies (b) and belongs to $\Pi(a \cap \lambda^+)$. The latter holds as there is a closed unbounded $C \subseteq \delta$, with $\operatorname{otp}(C) = \operatorname{cf}(\delta) < \operatorname{Min} a$, so $f^0_{\lambda,\alpha}(\theta) \leqq \bigcup_{\beta \in C} f^0_{\lambda,\beta}(\theta) < \theta$ as $f_{\lambda,\beta}(\theta) < \theta$ and $\operatorname{cf} \theta = \theta \geqq \operatorname{Min} a > |C|$. Then we can prove by induction on n, $f^n_{\lambda,\alpha}(\theta) < \theta$, and then $f_{\lambda,\alpha}(\theta) < \theta$.

For the first point (for $\beta < \alpha = \delta$, $f_{\lambda,\beta} <_J f_{\lambda,\delta}$) for every $\theta \in a \cap \lambda^+$, for some club C_θ of δ we have

$$(*) \quad f^0_{\lambda,\delta}(\theta) = \bigcup \{f_{\lambda,\beta}(\theta) : \beta \in C_\beta\}.$$

We can find $\gamma \in \bigcap_{\theta \in a \cap \lambda^+} C_\theta$, $\gamma > \beta$; by the induction hypothesis $f_{\lambda,\beta} <_J f_{\lambda,\gamma}$, whereas by $(*)\, f_{\lambda,\gamma} \leqq f^0_{\lambda,\delta}$. Trivially $f^n_{\lambda,\alpha} \leqq f^{n+1}_{\lambda,\alpha}$ so $f^0_{\lambda,\alpha} \leqq f_{\lambda,\alpha}$. Together we finish.

(2) By 7.4(3) for $\kappa = \aleph_0$ (see 7.2(4)).

(3) Let $\bar{b} = \langle b_\lambda : \lambda \in c \rangle$; and for each $\lambda \in c$ we can find $\alpha = \alpha_\lambda < \lambda$ such that $g \restriction b_\lambda < f_{\lambda,\alpha} \restriction b_\lambda \bmod J^{1,\kappa}_{<\lambda}$. Let $b^*_\lambda = \{\theta \in b_\lambda : g(\theta) < f_{\lambda,\alpha}(\theta)\}$, so $b^*_\lambda \subseteq b_\lambda$ and $b_\lambda \setminus b^*_\lambda \in J^{1,\kappa}_{<\lambda}$. If for some $d \subseteq c$, $|d| < \kappa$ and $a = \bigcup_{\lambda \in d} b^*_\lambda$, we are done; otherwise let J be the κ-complete filter generated by $\{b^*_\lambda : \lambda \in c\}$, let μ be minimal in c such that $J^{1,\kappa}_{<\mu^+}[a] \not\subseteq J$. Necessarily $\mu \in \operatorname{pcf}^{1,\kappa}(a) \subseteq c$, and choose $d \in J^{1,\kappa}_{<\mu^+}[a] - J$; so $d - b_\mu \in J^{1,\kappa}_{<\mu}[a] \subseteq J$ and $b_\mu - b^*_\mu \in J^{1,\kappa}_{<\mu}[a] \subseteq J$, together $d \in J$, contradiction.

7.5. Claim. Suppose

(a) $|a|^+ < \operatorname{Min} a$,

(b) $\bar{b} = \langle b_\theta : \theta \in c \rangle$ is a weak generating sequence for a,

(c) $\bar{f} = \langle \langle f_{\lambda,\alpha} : \alpha < \lambda \rangle : \lambda \in c \rangle$ is a $*$ continuous cofinality sequence for $(a, \bar{b})$,

(d) χ is large enough, $|a| < \sigma < \operatorname{Min} a$, $\sigma = \operatorname{cf}(\sigma)$, $N_i \prec (H(\chi), \in, <^*_\chi)$ for $i \leqq \sigma$, $N_i \in N_{i+1}$,

$$[i<j<\sigma \Rightarrow N_i \prec N_j], \quad a \in N_0, \quad \bar{f} \in N_0, \quad c \cup a \subseteq N_0,$$

$\| N_i \| < \operatorname{Min} a$, and for i limit $N_i = \bigcup_{j<i} N_j$,

(e) define $g_i \in \Pi a$ by $g_i(\theta) = \sup(N_i \cap \theta)$ (for $i \leqq \sigma$).

Then

(α) for $\lambda \in c$, $\delta \leqq \sigma$, $\operatorname{cf}(\delta) \in (|a|, \operatorname{Min} a)$ we have $f_{\lambda, g_\delta(\lambda)} \leqq g_\delta \restriction (a \cap \lambda^+)$,

(β) for $\lambda \in c$, $\delta \leqq \sigma$, $\operatorname{cf} \delta \in (|a|, \operatorname{Min} a)$ we have $f_{\lambda, g_\delta(\lambda)} \restriction b_\lambda = g_\delta \restriction b_\lambda \bmod J^0_{<\lambda}[a]$,

(γ) if $\bar{b}$ is a κ-almost generating sequence, $\delta \leqq \sigma$, $\operatorname{cf} \delta > |a|$, $c = \operatorname{pcf}^{1,\kappa}(a) = \operatorname{Dom} \bar{b}$, *then* for some $d \subseteq c$, $|d| < \kappa$ and $g_\delta = \operatorname{Max}\{f_{\lambda, g_\delta(\lambda)} : \lambda \in d\}$,

(δ) if $\langle b^\lambda_i : i_\lambda(*) \leqq i < \lambda \rangle \in N_0$ are as in $5.7(D)^+$ then (if $\delta \leqq \sigma$, $\operatorname{cf} \delta \in (|a|, \operatorname{Min} a)$)

$$d_\lambda \stackrel{\text{def}}{=} \{\theta \in a \cap \lambda^+ : f_{\lambda, g_\delta(\lambda)}(\theta) = g_\delta(\theta)\}$$

satisfies $d_\lambda \in J^0_{<\lambda^+}[a]$, $b^\lambda_{g_\delta(\lambda)} = d_\lambda \bmod J^0_{<\lambda}[a]$,

(ε) if $\lambda \in c$, $\delta \leqq \sigma$, $\operatorname{cf}(\delta) > |a|$, then $g_\delta \restriction b_\lambda$ is the $<_{J^0_{<\lambda}[\lambda]}$-lub of $\{f_{\lambda,\alpha} \restriction b_\lambda : \alpha < g_\delta(\lambda)\}$.

7.5A. Remark. (1) Using $J^{1,\kappa}_{<\kappa}[a]$ ($\lambda \in \operatorname{pcf}^{1,\kappa}(a)$) we have parallel results: if we restrict ourselves to $\operatorname{cf} \delta \in [\aleph_1, \kappa)$ the same continuity notion is O.K. (i.e. in addition to $\operatorname{cf}(\delta) \in [|a|^+, \operatorname{Min} a)$).

(2) For $\operatorname{cf} \delta = \aleph_0$, we should have a preassigned unbounded $C_\delta \subseteq \delta$, $\operatorname{otp} C_\delta = \omega$ for $\delta < \lambda$, $\operatorname{cf} \delta = \aleph_0$, and use $C \subseteq C_\delta$ in the definition of continuous.

Proof. Note that if $i < j \leqq \sigma$ then $g_i \in N_j$, so as $a \subseteq N_0$, $g_i < g_j$. As $\bar{f} \in N_0 \prec N_i$ and $a \subseteq N_0 \prec N_i$ for each $\theta \in a$, $g_i(\theta) \in N_j$ hence $f_{\theta, g_i(\theta)} \in N_j$ hence (as $\operatorname{Dom} f_{\theta, g_i(\theta)} = a \cap \lambda^+ \subseteq N_0 \prec N_j$) we have $\operatorname{Rang} f_{\theta, g_i(\theta)} \subseteq N_j$. By the definition of g_j this implies $f_{\theta, g_i(\theta)} \leqq g_j \restriction (a \cap \theta^+)$. Let $f^n_{\lambda,\alpha}$ ($\lambda \in c$, $n < \omega$, $\operatorname{cf} \alpha \in [|a|^+, \operatorname{Min} a)$) be as in 7.3.

Note that for $\theta \in a$, $\langle g_i(\theta) : i \leqq \sigma \rangle$ is strictly increasing continuous. So for limit $\delta \leqq \sigma$, $\operatorname{cf}(g_\delta(\theta)) = \operatorname{cf}(\delta)$, and $C_\theta \stackrel{\text{def}}{=} \{g_i(\theta) : i < \delta\}$ is a club of $g_\delta(\theta)$. So as $\bar{f}$ is $*$ continuous, if $\delta \leqq \sigma$, $|a| < \operatorname{cf}(\delta) < \operatorname{Min} a$, then $f^0_{\theta, g_\delta(\theta)}$ is defined by:

$$\text{for } \zeta \in a \cap \theta,\ f^0_{\theta, g_\delta(\theta)}(\zeta) = \operatorname{Min}\left\{ \bigcup_{\beta \in C} f_{\theta,\beta}(\zeta) : C \subseteq g_\delta(\theta) \text{ a club}\right\}.$$

Using C_θ we get

$$f^0_{\theta,g_\delta(\theta)}(\zeta) \leqq \bigcup_{\beta\in C_\theta} f_{\theta,\beta}(\zeta) = \bigcup_{i<\delta} f_{\theta,g_i(\theta)}(\zeta).$$

But we have noted above that $i<\delta \Rightarrow f_{\theta,g_i(\theta)} \leqq g_\delta \upharpoonright (a \cap \theta^+)$. So $f^0_{\theta,g_\delta(\theta)} \leqq g_\delta \upharpoonright (a \cap \theta^+)$. The same argument shows that if $\lambda \in C$, $\gamma<\lambda$, $\gamma \in \mathrm{cl}(\lambda \cap N_\delta)$ (closure in the order topology), $\delta \leqq \sigma$, $\mathrm{Min}\, a > \mathrm{cf}\,\delta > |a|$, *then* $\mathrm{Rang}\, f^0 \subseteq \mathrm{cl}(\lambda \cap N_\delta)$, noting

$$\mathrm{cf}\,\gamma \neq \mathrm{cf}\,\delta \,\&\, \gamma \in \mathrm{cl}(\lambda \cap N_\delta) \Rightarrow \gamma \in N_\delta \Rightarrow f^0_{\lambda,\gamma} \in N_\delta \Rightarrow \mathrm{Range}\, f^0_{\lambda,\gamma} \subset N_\delta,$$

so $\gamma \in \mathrm{cl}(\lambda \cap N_\delta) \Rightarrow \mathrm{Rang}\, f^0_{\lambda,\gamma} \subseteq N_\delta$. Now we can prove by induction on $\lambda \in C$ that

(*) $\delta \leqq \sigma$, $|a| < \mathrm{cf}\,\delta < \mathrm{Min}\,\alpha$, $\gamma \in \mathrm{cl}(\lambda \cap N_\delta)$, $n<\omega$;

we have $\mathrm{Rang}\, f^n_{\lambda,\delta} \subseteq \mathrm{cl}(N_\delta \cap \lambda)$ (this by induction on n); hence $\mathrm{Rang}\, f_{\lambda,\gamma} \subseteq \mathrm{cl}(\lambda \cap N_\delta)$. So we have proved ($\alpha$).

On the other hand, for each $\lambda \in c$, $i<j \leqq \sigma$, as $g_i \in (\Pi a) \cap N_j$, for some $\alpha = \alpha(\lambda, i)$ we have

$$\alpha \in N_j, \qquad g_i < f_{\lambda,\alpha} \bmod (J^0_{<\lambda}[a] + (a - b_\lambda)).$$

Now w.l.o.g., as $\alpha \in N_j$ we have $\alpha < g_j(\lambda)$, so

$$f_{\lambda,\alpha} < f_{\lambda,g_j(\lambda)} \bmod (J^0_{<\lambda}[a] + (a - b_\lambda)),$$

hence

$$g_i < f_{\lambda,g_j(\lambda)} \bmod (J^0_{<\lambda}[a] + (a - b_\lambda)).$$

So if $\delta \leqq \sigma$, $|a| < \mathrm{cf}\,\delta$, we have

$$g_i < f_{\lambda,g_\delta(\lambda)} \bmod (J^0_{<\lambda}[a] + (a - b_\lambda)) \quad \text{for each } i<\delta.$$

Let, for $i \leqq \delta$, $c_i \stackrel{\text{def}}{=} \{\theta \in a \cap \lambda^+ : g_i(\theta) > f_{\lambda,g_\delta(\lambda)}(\theta)\}$. Now as $[i<j \Rightarrow g_i \leqq g_j]$ we have $[i<j \Rightarrow c_i \subseteq c_j]$, so (as $\mathrm{cf}(\delta) > |a| = |\mathrm{Dom}\, g_i|$) $\langle c_i : i<\delta \rangle$ is eventually constant (by the definition of the c_j's and as $\langle g_j(\theta) : j \leqq \delta \rangle$ is increasingly continuous). As $c_\delta = \bigcup_{j<\delta} c_j$, so $c_\delta = c_i$ for some $i<\delta$. But we have shown above that for $i<\delta$, $c_i \in (J^0_{<\lambda}[a] + (a - b_\lambda))$; so $c_\delta \in J^0_{<\lambda}[a] + (a - b_\lambda))$, hence

$$\{\theta \in a \cap \lambda^+ : g_\delta(\theta) > f_{\lambda,g_\delta(\lambda)}(\theta)\} \in (J^0_{<\lambda}[a] + (a - b_\lambda)),$$

therefore

$$g_\delta \leqq f_{\lambda,g_\delta(\lambda)} \bmod (J^0_{<\lambda}[a] + (a - b_\lambda)).$$

As we have proved (α), if $\mathrm{cf}\,\delta \in (|a|, \mathrm{Min}\, a)$,

$$g_\delta \restriction b_\lambda = f_{\lambda, g_\delta(\lambda)} \bmod (J^0_{<\lambda}[a] + (a - b_\lambda)),$$

i.e. we get (β).

Now (γ), (ε) is left to the reader.

For a fixed λ let $g \in \Pi a$ be as in 5.7(D); w.l.o.g. $g \in N_0$. Let $d_\lambda \overset{\text{def}}{=} \{\theta \in a \cap \lambda^+ : g_\delta(a) = f_{\lambda, g_\delta(\lambda)}(\theta)\}$. By the definition of d_λ (as $g < g_\delta$ since $g \in N_\delta$) we have

$$\theta \in d_\lambda \Rightarrow g(\theta) < f_{\lambda, g_\delta(\lambda)}(\theta),$$

i.e. (noting that the minimal $i(*)$ satisfying 5.7(D)$^+$ belongs to N_0 and $i(*) + g_i(\lambda) = g_i(\lambda)$ for every i) by 5.7(D)$^+$

(*) $d_\lambda \cap (a \setminus b^\lambda_{g_\delta(\lambda)}) \in J^0_{<\lambda}[a]$, i.e, $d_\lambda \subseteq b^\lambda_{g_\delta(\lambda)} \bmod J^0_{<\lambda}[a]$.

On the other hand by (E) of 5.7 (and 5.5) certainly for every $\alpha < \delta$, $i \in \lambda \cap N_\alpha$, if $i \geqq i(*)$, then proof of (β) (of (7.5) holds also if we replace b_λ by b^λ_i, hence

$$f_{\lambda, g_\delta(\lambda)} \restriction b^\lambda_i = g_\delta \restriction b^\lambda_i \bmod J^0_{<\lambda}[a],$$

hence $b^\lambda_i \subseteq d_\lambda \bmod J^0_{<\lambda}[a]$.

To finish by (*) above we need just $b^\lambda_\delta \subseteq d_\lambda \bmod J^0_{<\lambda}[a]$; look at the proof of 5.7 and note:

7.6A. Subclaim. In 5.7, if $\langle f_i : i < \lambda \rangle$ is continuous (i.e. for $\delta < \lambda$, $|a| < \operatorname{cf} \delta < \operatorname{Min} a, f_\delta(\theta) = \operatorname{Min}\{\bigcup_{\alpha \in C} f_\alpha(\theta) : C \subseteq \delta \text{ a club}\}$, *then* for $d \subseteq a$, if $b_i \subseteq d \bmod J^0_{<\lambda}[a]$ for arbitrarily large $i < \delta$, then $b_\delta \subseteq d \bmod J^0_{<\lambda}[a]$.

Proof of 7.6A. Look at (iii) in the proof of 5.7.

7.6. Lemma. *Suppose* $|a| < \operatorname{Min} a$, $\bar{b} = \langle b_\lambda : \lambda \in a \rangle$ *is a weak generating sequence for* a.

Then we can find $\bar{b}' = \langle b'_\lambda : \lambda \in a \rangle, \bar{f} = \langle \langle f_{\lambda,\alpha} ; \alpha < \lambda \rangle : \lambda \in a \rangle$ *such that*:

(α) $\bar{b}'$ *is a smooth generating sequence*,

(β) *for* $\lambda \in a$, $b_\lambda \subseteq b'_\lambda \bmod J^0_{<\lambda}[a]$,

(γ) $\bar{f}$ *is a nice cofinality system*.

Proof. Let $\bar{f} = \langle \langle f^*_{\lambda,\alpha} : \alpha < \lambda \rangle : \lambda \in a \rangle$ be a $*$ continuous cofinality system for $(a, \bar{b})$. By 5.7 we can define $\langle b^\lambda_i : i_\lambda(*) < i < \lambda \rangle$, g^λ as there, satisfying (A)–(E) of 5.7. W.l.o.g. $i_\lambda(*) = 0$. We now define, by induction on $\lambda \in a$, $\langle f_{\lambda,\alpha} : \alpha < \lambda \rangle$. We define $f_{\lambda,\alpha}$ by induction on α such that:

(1) $f^*_{\lambda,\alpha+1} \leqq f_{\lambda,\alpha+1} \in \Pi(a \cap \lambda^+)$;

(2) for $\beta < \alpha, f_{\lambda,\beta} \restriction b_\lambda < f_{\lambda,\alpha} \restriction b_\lambda \bmod J^0_{<\lambda}[a]$;

(3) if $\alpha<\lambda$, $\operatorname{cf}\alpha \leqq |a|$ or $\operatorname{cf}(\alpha) \geqq \operatorname{Min} a$, we choose $f_{\lambda,\alpha}$ satisfying the relevant cases of (1) and (2) and, if possible,

(∗) $\theta \in \lambda \cap a \Rightarrow f_{\theta, f_{\lambda,\alpha}(\theta)} \leqq f_{\lambda,\alpha} \restriction (a \cap \theta^+)$;

(4) if $\alpha<\lambda$, $|a| < \operatorname{cf}\alpha < \operatorname{Min} a$, then $f^0_{\lambda,\alpha}(\theta) = \operatorname{Min}\{\bigcup_{\beta\in C} f_{\lambda,\beta}(\theta) : C$ a club of $\alpha\}$. $f^n_{\lambda,\alpha}, f_{\lambda,\alpha}$ are defined as in 7.3(2).

There are no problems in this.

Now choose χ large enough, $\sigma \overset{\text{def}}{=} |a|^+$ and $\langle N_i : i \leqq \sigma\rangle$ increasingly continuous, $N_i \prec (H(\chi), \in, <^*_\chi)$, $\|N_i\| = |a|^+$, $|a|^+ \subseteq N_i$, $N_i \in N_{i+1}$ and $\{\bar{f}, \langle\langle b^\lambda_i : i<\lambda\rangle : \lambda \in a\rangle, a\} \in N_0$. Now 7.5(α),(β) apply for $\delta=\sigma$, $\lambda \in a$ with b^λ_i for b_λ for any $i \in N_\sigma$. We can now show that in (3) above, (∗) was always possible: if not there is a minimal λ for which it fails and then a minimal α. So (λ,α) is definable from parameters which belong to N_0, hence $(\lambda,\alpha)\in N_0$. Now $g_\sigma \restriction (a \cap \lambda^+)$ shows (∗) is possible ($g_\sigma(\theta) \overset{\text{def}}{=} \sup(\theta \cap N_\sigma)$, of course). Moreover (∗) now holds also if $\alpha<\lambda$, $|a| < \operatorname{cf}(\alpha) < \operatorname{Min} a$ when $\operatorname{cf}[f_{\lambda,\alpha}(\theta)] = \operatorname{cf}\alpha$. So $\bar{f}$ is ∗ continuous and nice. Now let

$$b'_\lambda = \{\theta \in a \cap \lambda^+ : g_\sigma(\theta) = f_{\lambda, g_\sigma(\lambda)}(\theta)\};$$

they are as required.

§8. Kurepa trees from strong violation of GCH

8.1. Lemma. (1) *If* $\lambda \in \operatorname{pcf}(a)$, *every* $\lambda' \in \operatorname{pcf}(a)$, *is normal for* a *and for no inaccessible* μ, $\mu = |\operatorname{pcf}(a) \cap \mu|$, then *for some* $c \subseteq \lambda \cap \operatorname{pcf}(a)$ *with no last element*

$$\lambda = \operatorname{tcf}(\Pi c, <_{J_c^{\mathrm{bd}}}).$$

(2) *If* $\lambda \in \operatorname{pcf}(a)$, $\lambda = \max[\operatorname{pcf}(a)]$, $\sup \lambda \cap \operatorname{pcf}(a)$ *is singular*, then *for every unbounded* $c \subseteq \lambda \cap \operatorname{pcf}(a)$ *of power* $< \operatorname{Min} c$,

$$\lambda = \operatorname{tcf}(\Pi c, <_{J_c^{\mathrm{bd}}}).$$

Proof. (1) Find $b \in J_{\leq\lambda}[a] - J_{<\lambda}[a]$; by (2) we can find $c \subseteq \lambda \cap \operatorname{pcf}(b)$ as required.

Proof of 8.1(1). In more detail, the proof is by induction on $\mu = \sup[\lambda \cap \operatorname{pcf} a]$.

Case 1. In $\lambda \cap \operatorname{pcf}(a)$ there is no last element. So μ is a limit cardinal and cannot be inaccessible by a hypothesis. So μ is singular. We can find $c \subseteq \operatorname{pcf}(a) \cap \mu$, $|c| = \operatorname{cf}(\mu)(<\mu)$, $(\operatorname{cf}\mu)^+ < \operatorname{Min} c$.

By part (2) of 8.1, $\lambda = \operatorname{tcf} \Pi c / J_c^{\text{bd}}$.

Case 2. Not 1, so $\lambda \cap \operatorname{pcf}(a)$ has a last element κ say; so κ is normal for a, then b_κ^a is defined, and necessarily $\lambda \in \operatorname{pcf}(a \setminus b_\kappa^a)$; but $\kappa \notin \operatorname{pcf}(a \setminus b_\kappa^a)$, so if $\sup(\operatorname{pcf}(a) \cap \lambda) = \kappa$, we get Case 1, otherwise we use induction hypothesis on κ.

(2) By 5.12.

Proof of 8.1(2). Again the details are as follows: first $\max \operatorname{pcf}(c) \leqq \lambda$, as $\operatorname{pcf}(c) \subseteq \operatorname{pcf}(a)$ by 5.12. If $\neg\ [\operatorname{tcf}(\Pi c, <_{J_c^{\text{bd}}}) = \lambda)$, then $J_{<\lambda}[c] \not\subseteq J_c^{\text{bd}}$ (definitions), so for some $d \subseteq c$, $d \notin J_c^{\text{bd}}$ and $\theta \overset{\text{def}}{=} \max \operatorname{pcf}(d) < \lambda$.

Now $(\Pi d, <_{J_c^{\text{bd}}})$ is $\sup(d)$-directed, so $\theta \geqq \sup(d)$; $\sup d$ is singular, so $\sup d < \theta < \lambda$. Now $d \subseteq \operatorname{pcf}(a)$ and $|d| \leqq |c| < \operatorname{Min} c \leqq \operatorname{Min} d$, hence $\operatorname{pcf}(d) \subseteq \operatorname{pcf}(c)$ by 5.12, but $\theta \in \operatorname{pcf}(d)$ so $\theta \in \operatorname{pcf}(c)$. $\sup(\operatorname{pcf} a \cap \lambda) = \sup c = \sup d < \theta < \lambda$ — contradiction.

8.2. **Theorem.** *Suppose*:

(a) $\kappa = \operatorname{cf} \kappa > \aleph_0$,

(b) $\langle \mu_i^* : i < \kappa \rangle$ *is strictly increasing continuous*,

(c) $\mu_i^{**} = ((\mu_i^*)^\kappa)^+$ *is less than* μ_{i+1}^*,

(d) $\mu = \Sigma_{i<\kappa} \mu_i^*$,

(e) $\Sigma_{i<\kappa} |\operatorname{Reg} \cap (\mu_i^*, \mu_i^{**})| + |\operatorname{Reg} \cap (\mu, \mu^\kappa)| < \mu$.[†]

Then *we can find functions* $\langle h_\lambda : \lambda \in \operatorname{Reg} \cap (\mu, \mu^\kappa] \rangle$ *such that*:

(i) $\operatorname{Dom} h_\lambda = \kappa$;

(ii) $h_\lambda(i)$ *is a finite subset of* $\operatorname{Reg} \cap \bigcup_{j \leqq i} (\mu_j^*, \mu_j^{**})$;

(iii) *if* $\lambda \neq \theta$ *are from* $\operatorname{Reg} \cap (\mu, \mu^\kappa]$ *and* $i < \kappa$, *then*

$$h_\lambda(i) = h_\theta(i) \Rightarrow h_\lambda \restriction i = h_\theta \restriction i.$$

8.2A. **Remark.** (1) We ignore the possibility of exploiting "$|[\mu_i^*, \mu_i^{**}) \cap \operatorname{Reg}|$ is small for a stationary set of i's"; look at the proof and use Fodor's Lemma to do it.

(2) For $i < \kappa$ of cofinality $\aleph_0$ we can replace μ_i^{**} by

$$\operatorname{Min}\{\lambda : \text{for no } \lambda_j \in [\mu_j^*, \mu_j^{**}],\ \lambda > \max \operatorname{pcf}\{\lambda_j : j < i\}\}.$$

Proof. Let $a_i = \operatorname{Reg} \cap (\mu_i^*, \mu_i^{**})$, $a = \bigcup_i a_i$, $a_\kappa = \operatorname{Reg} \cap (\mu, \mu^\kappa]$, $a^* = a \cup a_\kappa$. By assumption (e), $|a| < \mu$, hence w.l.o.g. $|a| < \operatorname{Min} a$ and even

[†] Reg is the class of regular cardinals.

$(|a|^\kappa)^+ < \mathrm{Min}\, a$. By the Galvin–Hajnal theorem $|a_\kappa| < (|a|^\kappa)^+$, so $(|a^*|^\kappa)^+ < \mathrm{Min}\, a^*$. For each $\lambda \in a^*$ we can choose b_λ such that:

$(*)_a$ (i) $b_\lambda \subseteq a^* \cap \lambda^+$;
(ii) $b_\lambda \in J^0_{<\lambda^+}[a] - J^0_{<\lambda}[a]$;
(iii) $J^0_{<\lambda^+}[a] = J^0_{<\lambda}[a] + b_\lambda$

(use 7.2(3)).

Now by 7.6, w.l.o.g. $\langle b_\lambda : \lambda \in a^* \rangle$ is a smooth generating sequence. Note also that $\mathrm{pcf}(c) \subseteq \bigcup_{j \leq i} a_j$ for each $i < \kappa$ and $c \subseteq \bigcup_{j \leq i} a_j$ of cardinality $\leq \kappa$.

Now for each $\lambda \in \mathrm{Reg} \cap (\mu, \mu^\kappa]$, there is $c_\lambda \in [a]^\kappa$ such that $\lambda \in \mathrm{pcf}(c_\lambda)$ (see [Sh 111], 2.10† or [Sh 282], 12). By 5.8(3) w.l.o.g. $\lambda = \max \mathrm{pcf}(c_\lambda)$, hence $c_{\lambda_1} \neq c_{\lambda_2} \Leftrightarrow \lambda_1 \neq \lambda_2$. Let $c^*_\lambda = b_\lambda$, so $\lambda_1 \neq \lambda_2 \Leftrightarrow c^*_{\lambda_1} \cap \mu \neq c^*_{\lambda_2} \cap \mu$; so $\mathrm{pcf}(c^*_\lambda) = c^*_\lambda$. So for every $i < \kappa$,

$$\mathrm{pcf}\left(c^*_\lambda \cap \bigcup_{j \leq i} a_j\right) = c^*_\lambda \cap \bigcup_{j \leq i} a_j,$$

hence by 7.2(5) for some finite $d(\lambda, i) \subseteq c^*_\lambda \cap \bigcup_{j \leq i} a_j$, $\bigcup \{b_\theta : \theta \in d(\lambda, i)\} = c^*_\lambda \cap \bigcup_{j \leq i} a_j$. (We use smoothness.)

We can define h^*_λ; $h^*_\lambda(i) = d(\lambda, i)$.

8.3. Conclusion. If $2^{\aleph_1} < \aleph_{\omega_1}$, $i < \omega_1 \Rightarrow \aleph_i^{\aleph_1} < \aleph_{\omega_1}$ and $(\aleph_{\omega_1})^{\aleph_1} = \aleph_{\alpha(*)}$, $\alpha(*) \geqq \omega_2$, *then* there is an $\aleph_1$-Kurepa tree with $\geqq |\alpha(*)|$ branches.

Check (a)–(c) of 8.2, $\kappa = \aleph_1$, $\mu = \aleph_{\omega_1}$.

For the neophyte, the tree T is the following one:

The ith level is $T_i = \{h_\lambda \restriction (\mu^*_i, \mu^{**}_i) : \lambda \in \mathrm{Reg} \cap (\mu, \mu^\kappa)\}$;
the order is inclusion.

Clearly this is a tree with κ levels.

For $i < \kappa$, by (iii), $|T_i| \leqq |\{h_\lambda(i) : \lambda \in \mathrm{Reg}\}|$ which, by (ii), has power $\leqq \aleph_0 + |(\mu^*_j, \mu^{**}_j]|$, and for each $\lambda \in \mathrm{Reg} \cap (\mu, \mu^\kappa)$ let $\eta_\lambda = \langle h_\lambda \restriction i : i < \kappa \rangle$. η_λ is a κ-branch and clearly $h_{\lambda(1)} \neq h_{\lambda(2)} \neq \eta_{\lambda(1)} \neq \eta_{\lambda(2)}$, hence T has at least $|\mathrm{Reg} \cap (\mu, \mu^\kappa)|$ κ-branches.

† See paragraph before 2.8, and 2.8 which is from [GH]; there $\kappa = \omega_1$ is just for notational simplicity.

§9. Localizing pcf

9.1. CLAIM. Suppose $\langle a_i : i \leqq \kappa \rangle$ is increasing continuous, κ regular and $a = a_\kappa$ satisfies

$(*)_1$ $|\mathrm{pcf}(a)|^{\aleph_0} < \mathrm{Min}\, a$

or even just

$(*)_2$ there is a smooth generating sequence for $\mathrm{pcf}(a)$ and $|\mathrm{pcf}(a_\kappa)| < \mathrm{Min}\, a$.

(1) If $\lambda \in \mathrm{pcf}(a_\kappa) - \bigcup_{i<\kappa} \mathrm{pcf}(a_i)$ *then* for some $b \subseteq \bigcup_{i<\kappa} \mathrm{pcf}(a_i)$, $|b| \leqq \kappa$, $\lambda \in \mathrm{pcf}(b)$.

(2) If $\lambda \in \mathrm{pcf}(a_\kappa) - \bigcup_{i<\kappa} \mathrm{pcf}(a_i)$, $\kappa > \aleph_0$ *then*

$\oplus$ for some $S \subseteq \kappa$ unbounded, $\lambda_i \in \mathrm{pcf}(a_i) - \bigcup_{j<i} \mathrm{pcf}(a_j)$ for $i \in S$, we have $\mathrm{tcf}(\Pi_{i \in S} \lambda_i, <_{J_S^{\mathrm{bd}}}) = \lambda$, $\max \mathrm{pcf}\{\lambda_j : j < i\} < \lambda_i$.

9.1A. QUESTION. What about pcf^1?

9.1B. REMARK. In (2), we can waive the last demand but have S a club; see 9.3.

PROOF. (1), (2). Let $\bar{b} = \langle b_\theta : \theta \in \mathrm{pcf}(a_\kappa) \rangle$ be a generating sequence (exists: if $(*)_1$, by 6.14; if $(*)_2$, trivially). W.l.o.g. (by 5.8) $\lambda = \max \mathrm{pcf}(a_\kappa)$ and $\lambda \cap \mathrm{pcf}(a_\kappa)$ has no last element. By 7.6 w.l.o.g. $\bar{b}$ is smooth. By 7.2(5) for each i there is a finite $d_i \subseteq \mathrm{pcf}(a_i)$ such that:

(1) $\mathrm{pcf}(a_i) \subseteq \bigcup_{\theta \in d_i} b_\theta$, $|d_i| < \aleph_0$.

Let $d = \bigcup_{i<\kappa} d_i$, so $d \subseteq \bigcup_{i<\kappa} \mathrm{pcf}(a_i)$, $|d| \leqq \kappa$, so $\mathrm{Min}(d) > |d|$. If $\max \mathrm{pcf}(d) < \lambda$, then $d \in J^0_{<\lambda}[\mathrm{pcf}\, a_\kappa]$, hence for some finite $c \subseteq \mathrm{pcf}(d)$, $\mathrm{pcf}(d) \subseteq \bigcup_{\theta \in c} b_\theta$, hence $\bigcup_{i<\kappa} \mathrm{pcf}(a_i) \subseteq \bigcup_{\theta \in c} b_\theta$, but

$$\lambda = \max \mathrm{pcf}\, a_\kappa \leqq \max \mathrm{pcf}\Big(\bigcup_{i<\kappa} \mathrm{pcf}(a_i) \Big) \leqq \max \mathrm{pcf}\Big(\bigcup_{\theta \in c} b_\theta \Big)$$

$$\leqq \max_{\theta \in c} (\max \mathrm{pcf}(b_\theta)) = \max(c) < \lambda;$$

contradiction.

By the same proof we know that

(2) for any unbounded $S \subseteq \kappa$, $\lambda \in \mathrm{pcf} \bigcup_{i \in S} d_i$.

So $\max \mathrm{pcf}(d) \geqq \lambda$ but $\mathrm{pcf}(d) \subseteq \mathrm{pcf}(a_\kappa) \subseteq \lambda + 1$, so $\max \mathrm{pcf}(d) = \lambda$ which suffices for (1).

For (2) by Fodor's Lemma (note that κ is regular), so there are $\alpha < \kappa$, $n(*) < \omega$, and stationary $S \subseteq \kappa$ such that

$$d_i \cap \left(\bigcup_{j<i} \mathrm{pcf}(a_i) \right) \subseteq \mathrm{pcf}(a_\alpha),$$

$$\left| d_i - \bigcup_{j<i} \mathrm{pcf}(a_i) \right| \equiv n(*).$$

We now define by induction on $l \leqq n(*)$, S_l, $d_{i,l}$ such that:

(α) $S_0 = S$, $S_{l+1} \subseteq S_l$, $|S_l| = \kappa$,

(β) $d_{i,0} = d_i - \bigcup_{j<i} \mathrm{pcf}(a_j)$ for $i \in S_0$,

(γ) $d_{i,l+1}$ is a proper subset of $d_{i,l}$ for $i \in S_{l+1}$,

(δ) $\max \mathrm{pcf}(\bigcup \{d_{i,l} - d_{i,l+1} : i \in S_{l+1}\}) < \lambda$,

(ε) for all $i \in S_l$, $|d_{i,l}| = n_l$.

We continue till we are stuck; say $\langle d_{i,l} : i \in S_l \rangle$ are defined for $l \leqq m$, but not for $l = m+1$. By (δ)

$$\max \mathrm{pcf}(\bigcup \{d_i - d_{i,l} : i \in S_l\}) < \lambda \qquad \text{for } l \leqq m$$

(just prove it by induction on l, using (δ) and 5.3(2)). However, as said above (in (2)), $\lambda = \max \mathrm{pcf} \bigcup_{i \in S_l} d_i$, we conclude $\lambda = \max \mathrm{pcf}(\bigcup \{d_{i,l} : i \in S_l\})$, hence $d_{i,l} \neq \varnothing$ for $i \in S_l$, so $S_{n(*)}$ cannot be defined. If $\langle d_{i,m} : i \in S_m \rangle$ is last defined, $d^* = \bigcup_{i \in S_m} d_{i,m}$ satisfies almost all we need.

Now by the choice of m

$$c \subseteq d^* \ \& \ |c| = \kappa \Rightarrow \lambda = \max \mathrm{pcf}(c).$$

(Otherwise $S'_{m+1} = \{i \in S_m : c \cap d_{i,m} \neq \varnothing\}$ is unbounded in κ, hence for some unbounded $S_{m+1} \subseteq S'_{m+1}$, and n_{m+1}:

> $[i \in S_{m+1} \Rightarrow |d_{i,l} - c| = n'_{m+1}]$: now S_{m+1}, and $d_{i,m+1} \overset{\mathrm{def}}{=} d_{i,m} - c$ contradict the maximality of m.

On the other hand

$$c \subseteq d^* \ \& \ |c| < \kappa \Rightarrow (\exists i < \kappa) c \subseteq \mathrm{pcf}(a_i)$$
$$\Rightarrow \max \mathrm{pcf}(c) < \lambda.$$

We can easily make $\mathrm{pcf}(d^*) - \{\lambda\}$ have no last element and its sup minimal (replacing d^* by $d' \subseteq d$; $|d'| = \kappa$). But $\mathrm{pcf}(d^* \cap \mathrm{pcf}(a_i))$ has a last element (which is $< \lambda$), so $\langle \lambda_i \overset{\mathrm{def}}{=} \max \mathrm{pcf}(d^* \cap \mathrm{pcf}(a_i)) : i < \kappa \rangle$ is monotonic increasing and not eventually constant, and $\max \mathrm{pcf}\{\lambda_i : i < j\} < \sup\{\lambda_i : i < j\}$. So we have proved 9.1(2) too.

9.2. Claim. Suppose

$(*)_1$ $|\mathrm{pcf}(a)|^{\aleph_0} < \mathrm{Min}(a)$

or just

$(*)_2$ there is a generating sequence for pcf(a), and

$$|\mathrm{pcf}(a)| < \mathrm{Min}\, a.$$

If $b \subseteq \mathrm{pcf}(a)$, $\lambda \in \mathrm{pcf}(b)$, then for some $b' \subseteq b : |b'| \leqq |a|$, $\lambda \in \mathrm{pcf}(b')$.

Proof. We prove it by induction on $|b|$ and for a fixed $|b|$ by induction on λ. We can ignore the case "a is finite"; and w.l.o.g. $b = \max \mathrm{pcf}(a)$.

Case A: $|b| \leqq |a|$.

Trivial, let $b' = b$.

Case B: $|b| > |a|$.

Let $\kappa = \mathrm{cf}(|b|)$.

Let $\langle b_i : i < \kappa \rangle$ be increasingly continuous, $|b_i| < \kappa$, $b = \bigcup_{i<\kappa} b_i$. If for some $i < \kappa$, $\lambda \in \mathrm{pcf}(b_i)$, by the induction hypothesis there is $b' \subseteq b_i$ such that $\lambda \in \mathrm{pcf}(b')$, $|b'| \leqq \kappa$ and we finish. So w.l.o.g. for $i < \kappa$, $\lambda \notin \mathrm{pcf}(b_i)$. Now if $\kappa = \aleph_0$ we use 9.1(1): so there are $\lambda_n \in \mathrm{pcf}(b_n)$ for $n < \omega$ such that $\lambda \in \mathrm{pcf}\{\lambda_n : n < \omega\}$. By the induction hypothesis for each n for some $b'_n \subseteq b_n$, $|b'_n| \leqq |a|$ and $\lambda_n \in \mathrm{pcf}(b'_n)$. So $\bigcup_{n<\omega} b'_n$ is as required. So assume $\kappa > \aleph_0$. By 9.1(2) for some $\lambda_i \in \mathrm{pcf}(b_i)$, $\max \mathrm{pcf}\{\lambda_j : j < i\} < \lambda_i$, $\mathrm{tcf}(\Pi\lambda_i, <_{J^{\mathrm{bd}}_\kappa}) = \lambda$. For each $i < \kappa$, there is $b'_i \subseteq b_i$ such that $|b'_i| \leqq |a|$ and $\lambda_i \in \mathrm{pcf}(b'_i)$. If $|b|$ is singular, we have

$$\left| \bigcup_{i<\kappa} b'_i \right| \leqq \kappa + |a| = \mathrm{cf}(|b|) + |a| < |b|$$

and as $\lambda \in \mathrm{pcf}(\{\lambda_i : i < \kappa\}) \subseteq \mathrm{pcf} \bigcup_{i<\kappa} b'_i$, by the induction hypothesis on $|b|$ there is $b' \subseteq \bigcup_{i<\kappa} b'_i$, $\lambda \in \mathrm{pcf}(b')$, so we finish.

Hence w.l.o.g. $\kappa = |b|$, let $c_i = \{\lambda_j : j < i\}$. Let (see 7.6) $\langle b_\theta : \theta \in \mathrm{pcf}(a) \rangle$ be a smooth generating sequence for pcf(a). Let (by 7.2(5)) for each $i < \kappa$, d_i be a finite subset of pcf(c_i) such that $\mathrm{pcf}(c_i) \subseteq \bigcup_{\theta \in d_i} b_\theta$. Now $\langle \bigcup_{\theta \in d_i} b_\theta : i < \kappa \rangle$ is increasing (since for $i < j$, $d_i \subseteq \mathrm{pcf}(c_i) \subseteq \mathrm{pcf}(c_j) \subseteq \bigcup_{\theta \in d_j} b_\theta$ and $\tau \in \bigcup_{\theta \in d_j} b_\theta \Rightarrow b_\tau \subseteq \bigcup_{\theta \in d_j} b_\theta$) and hence so is $\langle a \cap (\bigcup_{\theta \in d_i} b_\beta) : i < \kappa \rangle$. As $\kappa > |a|$ the sequence is constant for $i \in [i(*), \kappa)$ for some $i(*) < \kappa$. But (remember that $\theta = \max b_\theta$ (trivially) hence $\max \mathrm{pcf}(\bigcup_{\theta \in d_i} b_\theta) = \max \mathrm{pcf}(c_i) = \max \bigcup_{\theta \in d_i} b_\theta = \max d_j$):

$$\max \operatorname{pcf}\left(a \cap \left(\bigcup_{\theta \in d_{i(*)+1}} b_\beta\right)\right) = \operatorname{Max} d_{i(*)+1}$$

$$= \max \operatorname{pcf}(c_{i(*)+1})$$

$$< \lambda_{i(*)+1}$$

$$\leqq \max \operatorname{pcf}\left(a \cap \left(\bigcup_{\theta \in d_{i(*)+1}} b_\theta\right)\right),$$

contradiction by the previous sentence.

9.3. LEMMA. *If* $(\forall \chi < \mu)(\chi^\kappa < \mu)$, $\operatorname{cf}(\mu) = \kappa > \aleph_0$, $\langle \mu_i : i < \kappa \rangle$ *is increasingly continuous*, $\bigcup_{i<\kappa} \mu_i = \mu$, $a_i = \operatorname{Reg} \cap (\mu_i, \mu_i^\kappa]$, $a = \bigcup_{i<\kappa} a_i$.

Then for any regular cardinal $\lambda \in (\mu, \mu^\kappa]$ *there is* $c_\lambda \subseteq a$, $|c_\lambda| = \kappa$, $\mu = \sup(c_\lambda)$ *such that* $\operatorname{tcf}(\Pi c_\lambda, <_{J^{bd}_{c_\lambda}}) = \lambda$ *and* $\{i < \kappa : c_\lambda \cap (\mu_i, \mu_i^\kappa] \neq \emptyset\}$ *is closed unbounded.*

PROOF. W.l.o.g. $\mu_0 > 2^\kappa$.

Let $b_i = \bigcup_{j \leqq i} a_j$, so $\langle b_i : i < \kappa \rangle$ is increasing, $b_i = \operatorname{pcf}(b_i)$. By [Sh 111], 2.10 for every $\lambda \in \operatorname{Reg} \cap (\mu, \mu^\kappa]$, $\lambda \in \operatorname{pcf}(c)$ for some $c \subseteq a$, $|c| \leqq \kappa$. Let $c_i^\lambda = \operatorname{pcf}(c) \cap b_i$, as $2^{|c|} \leqq 2^\kappa < \mu_0 < \operatorname{Min} c$, we can apply claim 9.1(2) to $\langle c_i^\lambda : i < \kappa \rangle$ to get $\langle \lambda_i : i < \kappa \rangle$. Now $c_\lambda \stackrel{\text{def}}{=} (\operatorname{pcf}\{\lambda_j : j < \kappa\}) \cap \mu$ is as required.

§10. Consistency of uniform copies of ω_1

10.1. THEOREM. $V \models$ *"$S = \{\kappa < \lambda : \kappa$ measurable$\}$ is stationary".* Then *for some semi-proper* P, $|P| = \lambda$, $P \models$ *κ-c.c. and*

> $\Vdash_P$ *"for every partition of $\mathscr{P}(\omega_1)$ to 2 there is a monochromatic homomorphic copy of ω_1 (in topology)".*

PROOF. We have $\diamondsuit_S$ w.l.o.g. We define by induction on $\alpha < \kappa$ a RCS iteration

$$\langle P_i \cdot \underset{\sim}{Q}_j : i \leqq \alpha, j < \alpha \rangle$$

such that

(*) each Q_j is semi-proper,

$$|P_i| \leqq \beth_{i+3}.$$

We know semi-properness is preserved (see [Sh A2], Ch. X, §2).

For most j, $Q_j = \text{Levi}(\aleph_1, 2^{2^{\aleph_1}})$. For $\kappa \in S$ we know that in V^{P_κ}

$(\oplus)$ $\forall$ countable $N \prec (H(\beth_8), \in)$ $\exists N'$ $N \prec N' \prec (H(\beth_8), \in)$ and $N \cap \omega_1 = N' \cap \omega_1$ and $\sup(N \cap \omega_2) < \sup(N' \cap \omega_2)$

(essentially see [Sh A2, Ch. XII, §2], strictly [Sh 253] 1.9, 1.9A(3)).

$\diamondsuit_S$ gives us a P_κ-name $\underset{\sim}{f} = \underset{\sim}{f}_\kappa$.

Assume $\Vdash_{P_\kappa} \underset{\sim}{f}: \mathscr{P}(\omega_1) \to \{\text{green, red}\}$ (otherwise use the usual Q_κ).

If in V^{P_κ} for f there is a homogeneous green set as required, do as usual: Levi collapse.

If not, let, in V^{P_κ}, $\mathscr{P} = \{A \subseteq \omega_1 : A \text{ non-stationary}\}$,

$$Q_\kappa = \{\langle A_i : i \leqq \alpha \rangle : A_i \subseteq \omega_1, A_i \in \mathscr{P} \text{ strictly increasing, continuous in } i \text{ and } f(A_i) = \text{red}\}$$

(the only properties of the family of non-stationary sets we use are: union of $\aleph_0$ is again in the family and is $\neq \omega_1$, and):

10.2. Claim. For N, N' from $\oplus$ necessarily in V^{P_κ},

$$\bigcup_{A \in N \cap \mathscr{P}} A \neq \bigcup_{A \in N' \cap \mathscr{P}} A.$$

Proof. For our iteration in V^{P_κ}, $2^{\aleph_1} = \aleph_2$. So $\mathscr{P} = \{B_\alpha : \alpha < \omega_2\}$. We can define $h : \omega_2 \to \mathscr{P}$,

$$h(\alpha) = \text{Min}\{\gamma : B_\gamma \text{ is not included in any union of countably many sets from } \{B_j : j < \alpha\}\}.$$

Easily h is well defined (even if $\neg$ CH)† and such $\langle B_\alpha : \alpha < \omega_2 \rangle$, h belong to N. Choose now $\alpha \in N' \cap \omega_2 \setminus N$. So $\bigcup_{A \in N \cap \mathscr{P}} A \not\supseteq A_{h(\alpha)} \subseteq \bigcup_{A \in N' \cap \mathscr{P}} A$ as

$$N \cap \mathscr{P} \subseteq \{B_\gamma : \gamma \in N \cap \omega_2\} \subseteq \{B_\gamma : \gamma < \alpha\}.$$

10.3. Claim. Q_κ is semi-proper (in V^{P_κ}).

Proof. Let $N \prec (H(\beth_8), \in)$ be countable, $p \in Q_\kappa \cap N$.

We can define (use $\oplus$ repeatedly) N_α $(\alpha < \omega_1)$ increasingly continuous, $N_\alpha \prec (H(\beth_8), \in)$, $N_\alpha \cap \omega_1 = N \cap \omega_1$, $\langle \sup(N_\alpha \cap \omega_2) : \alpha < \omega_2 \rangle$ strictly increasing. Now "$\bigwedge_{\alpha < \omega_1} f(\bigcup_{A \in \mathscr{P} \cap N_\alpha} A) = \text{green}$" is impossible as then $\langle f(\bigcup_{A \in \mathscr{P} \cap N_\alpha} A)$:

† By the diagonal union for some $B \in \mathscr{P}$, $[j < \alpha \Rightarrow B_j \setminus B$ countable$]$, $B_\gamma = \{i < \omega_1 : i \in B$ or $i = \sup(i \cap B)$ but $\text{otp}(i \cap B)$ not divisible by $\omega^2\}$.

$\alpha < \omega_1\rangle$ is a green set. So $\exists \alpha\, f(\bigcup_{A \in \mathscr{P} \cap N_\alpha} A) = \text{red}$. In N_α choose $p_n \in N \cap Q_\kappa$, $p_0 = p$, p_n increasing, $(\forall D \in N)$ [D dense subset of $Q_\kappa \rightarrow \bigvee_n p_n \in D$]. It is enough "$\bigcup p_n$ has a limit". Let $p_n = \langle B_\zeta : \zeta \leqq \alpha_n \rangle$, α_n increasing.

10.4. Claim. If $A \in \mathscr{P} \cap N_\alpha$ then $(\exists n) A \subseteq B_{\alpha_n}$.

Proof. $D_0 = \{\langle B'_\zeta : \zeta \leqq \alpha \rangle \in Q_\kappa : A \subseteq B'_\alpha\}$ is a dense subset of Q_κ: if $\langle B'_\zeta : \zeta \leqq \beta \rangle \in Q_\kappa$ also $(\exists X) \in \mathscr{P})(X \text{ red} \wedge X \supseteq B'_\zeta \cup A)$ — (if not we have a green cone), then $\langle B'_\zeta : \zeta \leqq \beta \rangle \hat{\ } \langle X \rangle \in D_0$. Now $D_0 \in N_\alpha$ so use definition of the p_n above.

Continuation of Proof of 10.3. By the claim

$$\bigcup_{\zeta < \bigcup_n \alpha_n} B_\zeta = \bigcup_{A \in \mathscr{P} \cap N_\alpha} A$$

which is red by choice of α. So $\langle B_\zeta : \zeta < \bigcup_n \alpha_n \rangle \hat{\ } \langle \bigcup_{A \in \mathscr{P} \cap N_\alpha} A \rangle$ is a limit of $\langle p_n : n < \omega \rangle$, belongs to Q_κ, so we have finished the proof of "Q_κ is semi-proper", hence of 10.1.

10.5. Remark. What about partitions of $\mathscr{P}_{<\aleph_1}(\aleph_2)$?

Velickovic and I discussed it in Arcta: from 2 colors, you cannot get rid of any; from 3, you can get rid of 1.

§11. On a problem of Archangelski

11.1. Example. (Answer q. 3 of Archangelski). Let λ be a cardinal. There is a space $X = X_\lambda$:

(1) with a basis of clopen sets (so it is a T_2 and T_3 space),

(2) $\Delta(X) = \psi(X) = \aleph_0$, i.e. in $X \times X$, the diagonal is the intersection of countably many open sets (hence every $x \in X$ has pseudo-character $\aleph_0$),

(3) cellularity $(X) = \aleph_0$,

(4) $|X_\lambda| = \lambda$.

11.1A. *Construction*. We define for $n < \omega$, $0 < m < \omega$ what is an m-place term $(0 < m < \omega)$ of depth $< n$, by induction on n (for such a term, $m = m[\tau]$, $n = n[\tau]$ are determined uniquely).

$n = 0$: it is a sequence $\tau = \langle 0, m \rangle$;

$n > 0$: for some terms $\tau_0, \ldots, \tau_{k-1}$ $(k < \omega)$, $n[\tau_i] < n$, and functions $h : \{0, \ldots, k-1\} \rightarrow \{1, -1\}$, $g : \{0, \ldots, k-1\} \rightarrow \{i : 0 < i < \omega\}$

and for $i<k$ strictly increasing functions $f_i:\{0,1,\ldots,m[\tau_i]-1\}\to\{0,1,\ldots,m-1\}$

such that (*) if $l_1, l_2<k$, $\tau_{l_1}=\tau_{l_2}$, $h(l_1)=1$, $h(l_2)=1$, then $g(l_1)=g(l_2)$.

Let $\tau=\langle n,m,\langle\tau_i:i<k\rangle,h,g,\langle f_i:i<k\rangle\rangle$ and we write $\tau_i=\tau_i[\tau]$, $h=h[\tau]$, $k=k[\tau]$, etc.

11.1B. *Observation.* The set of terms is countable.

11.1C. *The set of points.* Now the set of points of X_λ is

$$\{\langle\tau,\bar\alpha\rangle:\tau\text{ a term, }\bar\alpha\text{ an increasing sequence of ordinals }<\lambda\text{ of length }m[\tau]\}.$$

We write $\tau(\bar\alpha)$ instead of $\langle\tau,\bar\alpha\rangle$.

11.1D. *A basis and a pseudo nb basis for each point.* For each $0<l<\omega$ and $x\in X_\lambda$ we define sets u_x^l:

$$\begin{aligned}u_x^l=\{x\}\cup\{y:\ &\text{for some terms }\tau,\sigma\text{ and ordinals }\alpha_0<\cdots<\alpha_{m(\tau)-1}\\ &\text{we have }x=\tau(\langle\alpha_0,\ldots,\alpha_{m(\tau)-1}\rangle),\ y=\sigma(\beta_0,\ldots,\beta_{m(\sigma)-1})\\ &\text{and for some }i<k[\sigma]:\tau_i[\sigma]=\tau,\quad h[\sigma](i)=+1,\\ &[l<m(\tau)\ \&\ f_i[\sigma](l)=j\Rightarrow\alpha_l=\beta_j\text{ and }g[\sigma](i)\geqq l]\}.\end{aligned}$$

Note that $\oplus u_x^{l+1}\subseteq u_x^l$.

Now the topology of X_λ has the following base:

$$\left\{\bigcap_{i=0}^{p-1}[u_{x(i)}^{l(i)}]^{\varepsilon(i)}: p<\omega,\ x(i)\in X_\lambda,\ l(i)<\omega,\ \varepsilon(i)\in\{1,-1\}\text{ and}\right.$$
$$\left.[i,j<p,\ x(i)=x(j),\ \varepsilon(i)=1,\ \varepsilon(j)=-1\Rightarrow l(i)>l(j)]\right\}$$

where $u^1=u$, $u^{-1}=X_\lambda-u$ for $u\subseteq X_\lambda$.

11.1E. *Explanation.* We build the space like a free algebra. Each point x has a pseudo *nb* basis $\{u_x^l:n<\omega\}$, such that $u_x^{l+1}\subseteq u_x^l$, $\bigcap_{l<\omega}u_x^l=\{x\}$ (so $\psi(X_\lambda)=\aleph_0$); moreover

$$\bigcap_{l<\omega}\left(\bigcup_{x\in X_\lambda}u_x^l\times u_x^l\right)=\{(x,x):x\in X_\lambda\}.$$

We start with $\{\tau(\bar\alpha):n(\tau)=0\}$; the restriction to this set is the discrete topology. So (1) + (2) + (4) are O.K. For (3) (cellularity) we consider any finite intersection of u_x^l, $X_\lambda-u_x^l$ ($x=\tau(\bar\alpha)$, $n(\tau)=0$) for which there is no obvious reason why it should be empty; we add a point, i.e. an appropriate term exemplifying its non-emptiness. So two Boolean combinations of u^l_x's are not disjoint except when there is an obvious reason (e.g. u^8_x, $u^6_x - u^8_x$) and a point belongs to u^l_x only if it was added as a witness to an intersection including it.

11.1F. *Trivial properties*. Trivially $|X_\lambda| = \lambda$ (i.e. (4)) and X_λ has a basis of clopen sets (i.e. (1)).

11.1G. $\Delta(X_\lambda) = \aleph_0$. Suppose $x \neq y$ are from X_λ but $(x, y) \in \bigcap_l (\bigcup_z u^l_z \times u^l_z)$. So $x = \tau(\bar\alpha)$, $y = \sigma(\bar\beta)$. Let $l(*)$ be a natural number bigger than any $g[\tau](i)$ $(i < k[\tau])$, $g[\sigma](i)$ $(i < k[\sigma])$.

Now look at the definition of $u^{l(*)}_z$; clearly

$$x \in u^{l(*)}_z \Rightarrow x = z,$$

$$y \in u^{l(*)}_z \Rightarrow y = z.$$

As $y \neq x$, $(x, y) \notin \bigcup_z u^{l(*)}_z \times u^{l(*)}_z$.

11.1H. *Cellularity is* $\aleph_0$. Let $\{u_i : i < \omega_1\}$ be pairwise disjoint open non-empty subsets of X_λ. So as we can decrease them, w.l.o.g.

$$u_i = \bigcap_{p=0}^{q(i)-1} (u^{l(i,p)}_{x_{i,p}})^{\varepsilon(i,p)} \qquad \text{where } x_{i,p} \in X_\lambda.$$

As we can replace $\{u_i : i < \omega_1\}$ by any uncountable subfamily, w.l.o.g. $\varepsilon(i, p) = \varepsilon(p)$, $q(i) = q$, $l(i, p) = l(p)$ and for each p, $x_{i,p}$ $(i < \omega_1)$ are all equal or all distinct. Also w.l.o.g. the truth value of $x_{i,p_1} = x_{i,p_2}$ does not depend on i and

$$x_{i_1,p_1} = x_{i_2,p_2} \Rightarrow x_{i_1,p_1} = x_{i_2,p_2} = x_{i_1,p_2}.$$

Now we can easily form a $\tau(\bar\alpha)$ in $u_0 \cap u_1$.

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